I want to break symmetry!

Miriam Penners, Hannah Schäfer

August 6, 2024

Helmholtz-Institut für Strahlen- und Kernphysik, Universität Bonn

Outline

- Lagrangians
- Symmetries and group theory
- Particle physics, QCD, and effective field theories
- Chiral symmetry and (non)conserved Noether currents
- Spontaneous symmetry breaking and Goldstone bosons
- Chiral Lagrangian
- (Explicit symmetry breaking and low-energy constants)

Group theory

A **group** is a **set** $G = \{g_1, ..., g_N\}$ together with a **group operation**

$$\circ : G \times G \to G, \quad (g_i, g_j) \mapsto g_i \circ g_j = g_k, \quad i, j, k \in \{1, ..., N\}$$

with the following properties

- associativity: $(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k) \quad \forall g_i, g_j, g_k \in G$
- neutral element: $\exists e \in G : e \circ g = g = g \circ e \quad \forall g \in G$
- inverse elements: $\forall g \in G \;\; \exists g^{-1} \in G : g \circ g^{-1} = e = g^{-1} \circ g$

A group is called commutative/abelian if
$g_i \circ g_j = g_j \circ g_i \quad \forall g_i, g_j \in G$

Group actions and representations

Lie groups

- infinitely many elements, parameterised by parameter θ_i
- is also a differentiable manifold (think: line, surface,...)
- can write every group element $g \in G$

$$g = e^{\mathrm{i}\theta_i a_i}$$

- infinitely many elements, parameterised by parameter θ_i
- is also a differentiable manifold (think: line, surface,...)
- can write every group element $g \in G$

$$g = e^{\mathrm{i}\theta_i a_i}$$

- a_i **group generators**, form LIE algebra $\mathfrak{g}$ for the LIE group G
- (algebra: vector space (with addition, scalar multiplication) with LIE bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (a_i, a_j) \mapsto [a_i, a_j] = a_k$)

Important Lie groups in physics

- $(S)O(n)$ ((special) orthogonal group, rotations: $O^{\mathsf{T}}O = \mathbb{1}$)
- $(S)U(n)$ ((special) unitary group: $U^{\dagger}U = \mathbb{1}$)
  $\rightarrow$ dim $SU(n) = n^2 - 1$

$$U(n) \simeq SU(n) \times U(1), \quad g \in U(1) \Rightarrow g = e^{i\theta \cdot 1}$$

("it's just a phase")

- generators of $SU(3)$: Gell-Mann matrices $\{\lambda_j\}_{j=1}^{8}$

Particle physics

Associate **particles** with **quantised field** $\phi_1(x), \phi_2(x), \ldots$ at position $x = (t, \vec{x})$

- $\phi_i(x)$ contain creation & annihilation operators acting on states
- particles ordered according to **quantum numbers**: behaviour under different symmetry operators
  - LORENTZ group (space-time symmetries: rotations, translations, boosts), spin S
  - discrete symmetries, e.g., parity $\phi(x) \xrightarrow{P} +\phi(x)/-\phi(x)$

$\Rightarrow$ different representations of the LORENTZ group

S/P	$+$	$-$
0	scalar	pseudoscalar
1	axial vector	vector

Build Lagrangian from fields and couplings, terms like $m^2\phi_1(x)\phi_1(x)$, $c\,\phi_1(x)\phi_1(x)\phi_2(x)$, ...

- SM includes electromagnetic (em) interaction ($U(1)$), weak interaction ($SU(2)$), and strong interaction ($SU(3)$) with couplings $\alpha_{\mathrm{em}}, \alpha_{\mathrm{w}}$ (actually a bit more involved), and α_s
- NOETHER current in 4-d: j^μ, **charge**: $Q = \int \mathrm{d}^3x\, j^0$
- calculate decay or scattering rates via **Feynman rules**: everything that can happen will happen, sum over all processes $\Rightarrow$ infinite series in coupling constants, perturbative QFT

Strong Interaction: Quantum Chromodynamics

- additional symmetry/quantum number: isospin (proton/neutron), $SU(2)$
- more generally: "flavour", quarks come in $n_f = 6$ different flavours, u, d, s, c, b, t, $SU(n_f)$
- gauge bosons: gluon, couples to colour-charged particles (quarks and gluons), $SU(n_c)$

$$\mathcal{L}_{\text{QCD}} = \sum_{f \in \{u,d,s,c,b,t\}} \bar{q}_f \Big(i \underbrace{\not{D}}_{\text{covariant derivative}} - \underbrace{\mathcal{M}_f}_{\text{quark masses}} \Big) \underbrace{q_f}_{\text{quark field}} - \underbrace{\frac{1}{4} G_{\mu\nu,a} G^{\mu\nu,a}}_{\text{kinetic term of gluons}}$$

The Problem with $\mathcal{L}_{\text{QCD}}$

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{q}_f \left(i\not{D} - \mathcal{M}_f \right) q_f - \frac{1}{4} G_{\mu\nu,a} G^{\mu\nu,a}$$

Problem: at low energies no perturbation theory possible
$\rightarrow$ construct low-energy **effective** field theory (EFT)

- Find relevant **degrees of freedom** (dofs)
  Billiards: neglect finite mass of border and deformations, instead scattering of balls off "infinitely heavy" border
- Find allowed terms of Lagrangian $\mathcal{L}_{\chi\text{PT}}$
  $\mathcal{L}_{QCD}$ and $\mathcal{L}_{\chi PT}$ must have same symmetries

$\rightarrow$ **Symmetries** of $\mathcal{L}_{\text{QCD}}$ needed for finding both!

Symmetries of $\mathcal{L}_{\text{QCD}}$

What symmetries does $\mathcal{L}_{\text{QCD}}$ possess?

- Lorentz invariance
- discrete symmetries P, C, T
- $SU(3)_c$ gauge invariance by design

What else?

Low-energy EFT: consider only light quarks (u, d, s)

$\rightarrow$ find approximate **chiral symmetry**

What is that?

Chiral Symmetry – Projection Operators

- define **projection operator** $P_{L/R} = \frac{1}{2}(\mathbb{1} \mp \gamma_5)$
- decomposes quark fields into left- and right-handed chiral components,

$$q_L = P_L q \text{ and } q_R = P_R q \text{ with } q = q_L + q_R$$

(*"*LORENTZ *invariant version of handedness"*)

- P_L and P_R are projection operators, because ...
  - $P_L^2 = P_L$ and $P_R^2 = P_R$
  - $P_L P_R = 0 = P_R P_L$
  - $P_L + P_R = \mathbb{1}$

use $\gamma_5^2 = \mathbb{1}$

Chiral Symmetry – Chiral Decomposition

$$\mathcal{L}_{\text{QCD}} = \underbrace{-\frac{1}{4}G_{\mu\nu,a}G^{\mu\nu,a} + \bar{q}i\not{D}q}_{\mathcal{L}^0_{\text{QCD}}} - \underbrace{\bar{q}\mathcal{M}q}_{\mathcal{L}^m_{\text{QCD}}}$$

Decomposition of the quark terms yields

- $\bar{q}\not{D}q = \bar{q}_L\not{D}q_L + \bar{q}_R\not{D}q_R$
- $\bar{q}\mathcal{M}q = \bar{q}_R\mathcal{M}q_L + \bar{q}_L\mathcal{M}^\dagger q_R \to$ couples q_L and q_R

using $\gamma_5^\dagger = \gamma_5$ and $\{\gamma_5, \gamma_0\} = \gamma_5\gamma_0 + \gamma_0\gamma_5 = 0$
$\Rightarrow$ In **chiral limit** ($m_u = m_d = m_s = 0$), $\mathcal{L}^0_{\text{QCD}}$ invariant under chiral $U(3)_L \times U(3)_R$ flavour transformations

Chiral Symmetry – Flavour transformations

$\mathcal{L}^0_{\text{QCD}}$ invariant under independent $U(3)$ transfos of q_L and q_R

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i \sum_{j=1}^{8} \Theta_j^L \frac{\lambda_j}{2}\right) e^{-i\Theta^L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}$$

$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp\left(-i \sum_{j=1}^{8} \Theta_j^R \frac{\lambda_j}{2}\right) e^{-i\Theta^R} \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

- acting in flavour space (ie. u, d, s)
- *global* and *continuous* symmetry!
- decomposed into $U(3) = SU(3) \times U(1)$
  8 Gell-Mann matrices & 1 phase factor → still 9 generators

Chiral Symmetry – Noether Currents

- Rewrite whole symmetry group

$$U(3)_L \times U(3)_R = SU(3)_L \times SU(3)_R \times U(1)_L \times U(1)_R$$

- Obtain Noether currents

$$L_0^\mu = \bar{q}_L \gamma^\mu q_L, \quad L_j^\mu = \bar{q}_L \gamma^\mu \frac{\lambda_j}{2} q_L$$
$$R_0^\mu = \bar{q}_R \gamma^\mu q_R, \quad R_j^\mu = \bar{q}_R \gamma^\mu \frac{\lambda_j}{2} q_R$$

using $j^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu q)} \delta q$ with $q \mapsto q + \alpha \delta q$

Chiral Symmetry – Group Theory Shenanigans

- introduce $V = L + R$ and $A = R - L$
  compare to 2-body system with equal masses:
  center of mass $\vec{R} = \frac{1}{2}(\vec{x} + \vec{y})$ *and relative coordinate* $\vec{r} = \vec{x} - \vec{y}$
- again rewrite whole symmetry group

$$U(3)_L \times U(3)_R = SU(3)_L \times SU(3)_R \times U(1)_L \times U(1)_R$$
$$= SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A$$

- obtain **vector**- and **axial-vector currents**

$$V_0^\mu = \bar{q}\gamma^\mu q, \qquad V_j^\mu = \bar{q}\gamma^\mu \frac{\lambda_a}{2} q$$
$$A_0^\mu = \bar{q}\gamma^\mu \gamma_5 q, \quad A_j^\mu = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda_a}{2} q$$

Chiral Symmetry – Currents Conserved?

Are these currents conserved, $\partial_\mu j^\mu = 0$? (consider quark masses to assess explicit symmetry breaking)
Use free Dirac equation $\partial\!\!\!/ q = -i\mathcal{M}q$ and $\bar{q}\overleftarrow{\partial\!\!\!/} = \bar{q}i\mathcal{M}$

- $\partial_\mu V_0^\mu = 0$ for any $\mathcal{M}$
- $\partial_\mu A_0^\mu = 2i\bar{q}\mathcal{M}\gamma_5 q +$ quantum corrections $\rightarrow \neq 0$ even for $\mathcal{M} = 0$, symmetry only conserved on classical level
- $\partial_\mu V_j^\mu = i\bar{q}\left[\mathcal{M}, \frac{\lambda_j}{2}\right] q$
- $\partial_\mu A_j^\mu = i\bar{q}\left\{\mathcal{M}, \frac{\lambda_j}{2}\right\} q$

$\rightarrow$ remaining symmetry group of $\mathcal{L}^0_{\text{QCD}}$ ($\mathcal{M} = 0$):

$$SU(3)_V \times SU(3)_A \times U(1)_V$$

Charges and Generators

- Calculate charges of V_j and A_j

$$Q_{V,j} = \int \mathrm{d}^3x V_j^0(\vec{x},t), \quad Q_{A,j} = \int \mathrm{d}^3x A_j^0(\vec{x},t)$$

- from $\partial_\mu j^\mu = 0$ we see that **charges conserved**, $\partial_t Q = 0$
- Q commutes with Hamiltonian H
- Q associated with generator of the symmetry

Spontaneous symmetry breaking

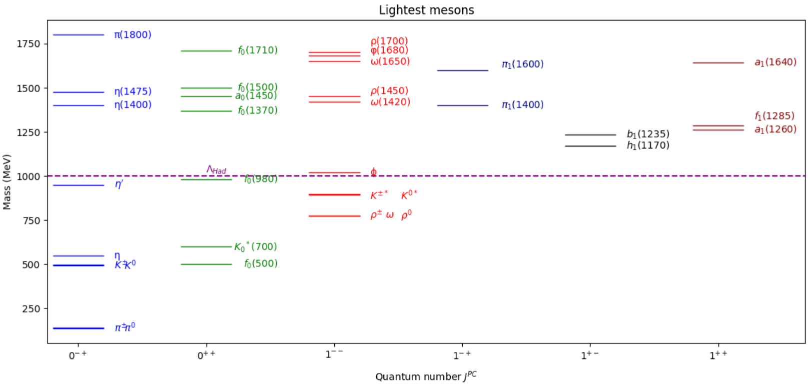
- a symmetry can manifest itself in different ways: the conserved charge always commutes with the Hamiltonian, $[Q, H]$, but it can either
  - annihilate the vacuum, $Q\,|0\rangle = 0$, Wigner-Weyl mode
  - not annihilate the vacuum, $Q\,|0\rangle \neq 0$, Nambu-Goldstone mode

    $\Rightarrow$ vacuum not invariant, emergence of massless excitations: Goldstone bosons
- Wigner-Weyl mode for V_j and A_j would imply "parity doubling" in hadronic spectrum

  $\Rightarrow$ do not see this, hence Nambu-Goldstone mode is realised for A_j, **spontaneous symmetry breaking**

Pseudo-Goldstone bosons

- Low energies, **8 pseudo-Goldstone bosons** are relevant dofs!
  $\rightarrow$ identify as π^+, π^-, π^0, K^+, K^-, K^0, $\bar{K}^0$, η
- Why "pseudo"?
  Since quark masses $\neq 0$: π, K and η *do have mass!*



Spectrum of light hadrons (thanks to Leon Heuser!)

Constructing $\mathcal{L}_{\chi\text{PT}}$ – Describing Fields

How to construct effective Lagrangian describing these fields?
$\rightarrow$ more group theory!

Short answer:

- fields contained in unitary matrix

$$U = \exp\left\{\frac{i}{F}\lambda_j\Phi_j\right\} = \exp\left\{\frac{\sqrt{2}i}{F}\begin{pmatrix} \frac{\Phi_3}{\sqrt{2}} + \frac{\Phi_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\Phi_3}{\sqrt{2}} + \frac{\Phi_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\Phi_8}{\sqrt{6}} \end{pmatrix}\right\}$$

- U transforms under
  $G = SU(3)_R \times SU(3)_L = \{(R, L) | R \in SU(3), L \in SU(3)\}$
  according to

$$U \overset{G}{\mapsto} \tilde{U} = LUR^\dagger$$

- Construct Lagrangian, s.t. invariant under group action of G

Constructing $\mathcal{L}_{\chi\text{PT}}$ – Describing Fields 2

Long answer:

- $G = SU(3)_R \times SU(3)_L = \{(R, L) | R \in SU(3), L \in SU(3)\}$ is symmetry group of $\mathcal{L}^0_{\text{QCD}}$
- **unbroken subgroup** $H = \{(V, V) | V \in SU(3)\}$
- $\exists$ isomorphic mapping between quotient group $G/H = \{gH | g \in G\}$ (left coset) and the GOLDSTONE boson fields
- properties of φ for $g_1, g_2 \in G$:

$$\varphi(e, \Phi) = \Phi \quad \text{and} \quad \varphi(g_1, \varphi(g_2, \Phi)) = \varphi(g_1 g_2, \Phi)$$

- with $\Phi = \varphi(f, 0)$, $f = gh \in gH$, 0 as "ground state", we find

$$\varphi(\tilde{g}, \Phi) = \varphi(\tilde{g}, \varphi(gh, 0)) = \varphi(\tilde{g}gh, 0) =: \varphi(\tilde{f}, 0) = \tilde{\Phi}$$

- write $g = (R, L) \in G$ and
  $gH = (R, L)(V, V) = (RV, LV) = \cdots = (1, LR^\dagger)H$

$\rightarrow$ left coset can be uniquely defined by $U = LR^\dagger$ (unitary)

- How does it transform?

$$\tilde{g}gH = (\tilde{R}, \tilde{L})(1, U)H = (\tilde{R}, \tilde{L}U)H = (1, \tilde{L}U\tilde{R}^\dagger)H$$

$\rightarrow$ $U \overset{G}{\mapsto} \tilde{U} = \tilde{L}U\tilde{R}^\dagger$

Constructing $\mathcal{L}_{\chi\mathsf{PT}}$ – General Considerations

What can we do with U?

- "Dagger it": $U^\dagger = U^{-1}$
- take derivatives: $\partial_\mu U$

Can construct terms now! They need to be

- Lorentz invariant
  even number of derivatives (factors of momentum) per term;
  low momenta $\rightarrow$ # derivatives provide ordering scheme!
  Structure $\mathcal{L}_{\chi\mathsf{PT}}$ in terms of # derivatives

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots$$

- all possible allowed terms invariant under G
  $\rightarrow$ important tool: Trace! (Cyclicity: $\langle ABC \rangle = \langle CAB \rangle$)

Constructing $\mathcal{L}_{\chi\text{PT}}$ – Finding terms

- $\mathcal{L}_0$:

Constructing $\mathcal{L}_{\chi\mathsf{PT}}$ – Finding terms

- $\mathcal{L}_0$:
  - $\rightarrow$ need U and $U^\dagger$
  - $\rightarrow$ but $UU^\dagger = \mathbb{1} \rightarrow$ constant!

Constructing $\mathcal{L}_{\chi\mathsf{PT}}$ – Finding terms

- $\mathcal{L}_0$:
  - $\rightarrow$ need U and $U^\dagger$
  - $\rightarrow$ but $UU^\dagger = \mathbb{1} \rightarrow$ constant!
- $\mathcal{L}_2$:

Constructing $\mathcal{L}_{\chi\mathsf{PT}}$ – Finding terms

- $\mathcal{L}_0$:
  - $\rightarrow$ need U and $U^\dagger$
  - $\rightarrow$ but $UU^\dagger = \mathbb{1} \rightarrow$ constant!
- $\mathcal{L}_2$:
  - $\rightarrow$ find as only contribution $\langle \partial_\mu U \partial^\mu U^\dagger \rangle$
  - $\rightarrow$ not $\langle U \partial_\mu \partial^\mu U^\dagger \rangle$, because connected to above via integration by parts

Constructing $\mathcal{L}_{\chi\mathrm{PT}}$ – Finding terms

- $\mathcal{L}_0$:
  - $\rightarrow$ need U and $U^\dagger$
  - $\rightarrow$ but $UU^\dagger = \mathbb{1} \rightarrow$ constant!
- $\mathcal{L}_2$:
  - $\rightarrow$ find as only contribution $\langle \partial_\mu U \partial^\mu U^\dagger \rangle$
  - $\rightarrow$ not $\langle U \partial_\mu \partial^\mu U^\dagger \rangle$, because connected to above via integration by parts
- $\mathcal{L}_4$:
  - $\rightarrow$ $\langle \partial_\mu U \partial^\mu U^\dagger \rangle^2$
  - $\rightarrow$ $\langle \partial_\mu U \partial_\nu U^\dagger \rangle \langle \partial^\mu U \partial^\nu U^\dagger \rangle$
  - $\rightarrow$ $\langle \partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger \rangle$

Constructing $\mathcal{L}_{\chi\mathsf{PT}}$ – Where are the fields?

- Remember $U = \exp\{i\Phi/F\}, \quad \Phi = \lambda_j \Phi_j$
- How to get Lagrangian in terms of meson fields in the exponential?
- Expand!

$$U = \exp\left\{\frac{i}{F}\Phi\right\} \approx \mathbb{1} + \frac{i}{F}\Phi - \frac{1}{2F^2}\Phi^2 + \ldots$$

→ terms with four fields can stem from e.g. $\mathcal{L}_4$ or $\mathcal{L}_2$ at higher order in expansion of U

Explicit symmetry breaking: masses

Resources

Chiral Perturbation theory

- S. Scherer, M.R. Schindler: A Primer for Chiral Perturbation Theory, *Springer 2012*, doi:10.1007/978-3-642-19254-8.
- S. Scherer, M.R. Schindler: A Chiral perturbation theory primer, *arXiv 2005*, arXiv:hep-ph/0505265.
- B. Kubis: An Introduction to chiral perturbation theory, *Workshop on Physics and Astrophysics of Hadrons and Hadronic Matter 2007*, arXiv:hep-ph/070327.

Basics and further reading

- U.-G. Meissner, A. Rusetsky: Effective Field Theories, *Cambridge University Press 2022*, doi:10.1017/9781108689038.
- M.D. Schwartz: Quantum Field Theory and the Standard Model, *Cambridge University Press 2013*, doi:https:/10.1017/9781139540940.
- & many more books on QFT and group theory