Lissajous figures in a quantum walk on a lattice

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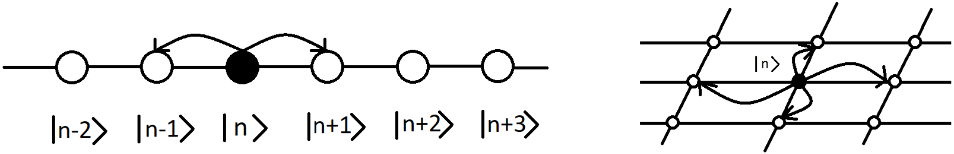
The Quantum Model



- Particle may occupy only discrete lattice sites
- Quantum state $|n\rangle$ $(n \in \mathbb{Z})$ describes a particle in n-th lattice site, $\langle n|m\rangle = \delta_{nm}$
- Hopping is permitted only between nearest neighbours
- State of a particle at any time can be written as $|\psi(t)\rangle = \sum_n \psi_n(t)\,|n\rangle$
- In 2D case, the diagonal hopping is not allowed

Hamiltonian

In that case system is described by the Hamiltonian with matrix elements given by:

$$\langle n | \hat{H} | m \rangle = \begin{cases} \epsilon_n & \text{for n=m} \\ -J & \text{if n,m are nearest neighbours} \\ 0 & \text{rest} \end{cases}$$

ϵ_n - energy of a particle in *n*-th site

J - tunnelling amplitude

Initial state

In my work I consider evolution of quantum states, that are Gaussian at the initial moment

$$\psi_n(0) = A \exp\left[-\frac{|\vec{r}_n - \vec{r}_0|^2}{4\sigma^2} + i\vec{k}_0 \cdot \vec{r}_n\right]$$

$$\sum_n |\psi_n|^2 = 1$$

$\vec{r}_0$ - center of a wavepacket
$\vec{k}_0$ - average momentum of a wavepacket

1D Case - free particle

The Hamiltonian is reduced to:

$$\hat{H} = -J \sum_j (|j\rangle \langle j+1| + |j+1\rangle \langle j|)$$

The Hamiltonian can be analytically diagonalized:

$$\hat{H} = \int_{-\pi}^{+\pi} \lambda_\kappa |\kappa\rangle \langle\kappa| \, d\kappa$$

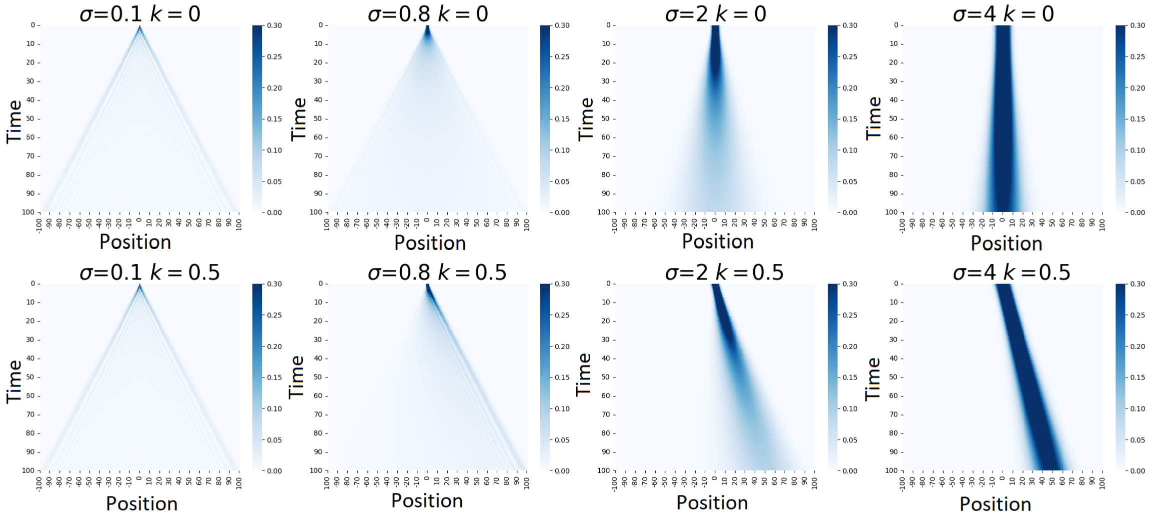
$$|\kappa\rangle = \frac{1}{\sqrt{2\pi}} \sum_j e^{i\kappa j} |j\rangle$$

$$\lambda_\kappa = -2J \cos\kappa$$

Then we obtain expression for a time evolution straightforwardly :

$$\psi_m(t) = A \sum_n \exp\left[-\frac{n^2}{4\sigma^2} + i\left(nk + \frac{\pi}{2}m - \frac{\pi}{2}n\right)\right] \mathcal{J}_{m-n}\left(\frac{2J}{\hbar}t\right)$$

$|\psi_m(t)|^2$



Time is given in units $\hbar/J$

1D Case - constant force

In that case a particle is subjected to linear external potential. The Hamiltonian in such case is:

$$\hat{H} = -J \sum_j (|j\rangle \langle j+1| + |j+1\rangle \langle j|) + \widetilde{F} \sum_j j |j\rangle \langle j|$$

Again, the Hamiltonian can be diagonalized:

$$\hat{H} = \sum_{\mathfrak{n}} \lambda_{\mathfrak{n}} |\mathfrak{n}\rangle \langle \mathfrak{n}|$$

$$|\mathfrak{n}\rangle = \sum_j \mathcal{J}_{j-\mathfrak{n}} \left(\frac{2J}{\widetilde{F}} \right) |j\rangle$$

$$\lambda_{\mathfrak{n}} = \mathfrak{n} \widetilde{F}$$

and one obtains time evolution:

$$\psi_m(t) = A \sum_n \exp \left[-\frac{n^2}{4\sigma^2} + i \left(nk + \frac{\pi}{2} m - \frac{\pi}{2} n - \frac{\widetilde{F}}{2\hbar} (m+n) t \right) \right] \mathcal{J}_{m-n} \left[\frac{4J}{\widetilde{F}} \sin \left(\frac{\widetilde{F}}{2\hbar} t \right) \right]$$

1D Case - constant force

$$\psi_m(t) = A\sum_n \exp\left[-\frac{n^2}{4\sigma^2} + i\left(nk + \frac{\pi}{2}m - \frac{\pi}{2}n - \frac{\widetilde{F}}{2\hbar}(m+n)t\right)\right]\mathcal{J}_{m-n}\left[\frac{4J}{\widetilde{F}}\sin\left(\frac{\widetilde{F}}{2\hbar}t\right)\right]$$

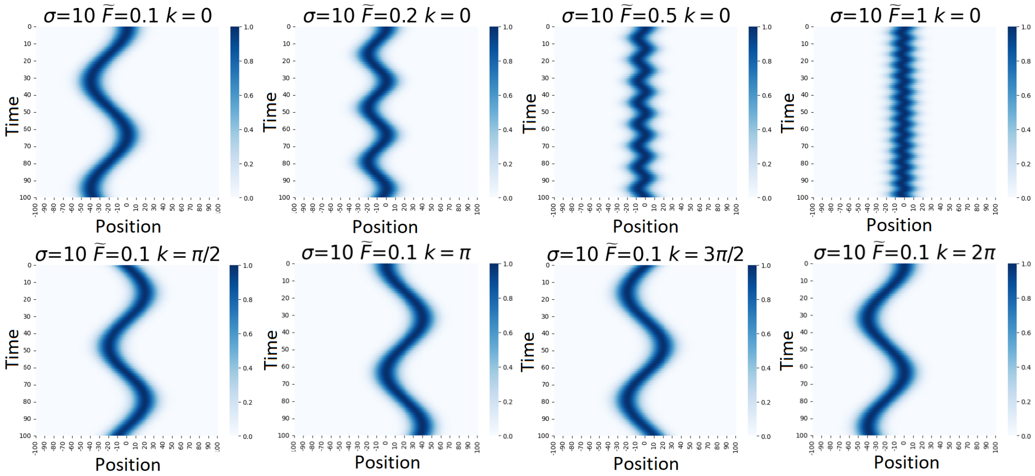
For wide wavepackets we can approximate that expression by

$$\psi_m(t) = A\exp\left[-\frac{(m-\langle m(t)\rangle)^2}{4\sigma^2} + im\left(k - \frac{\widetilde{F}}{\hbar}t\right) + i\Phi(t)\right],$$

$$\langle m(t)\rangle = \frac{2J}{\widetilde{F}}\left[\cos\left(\frac{\widetilde{F}}{\hbar}t - k\right) - \cos(k)\right].$$

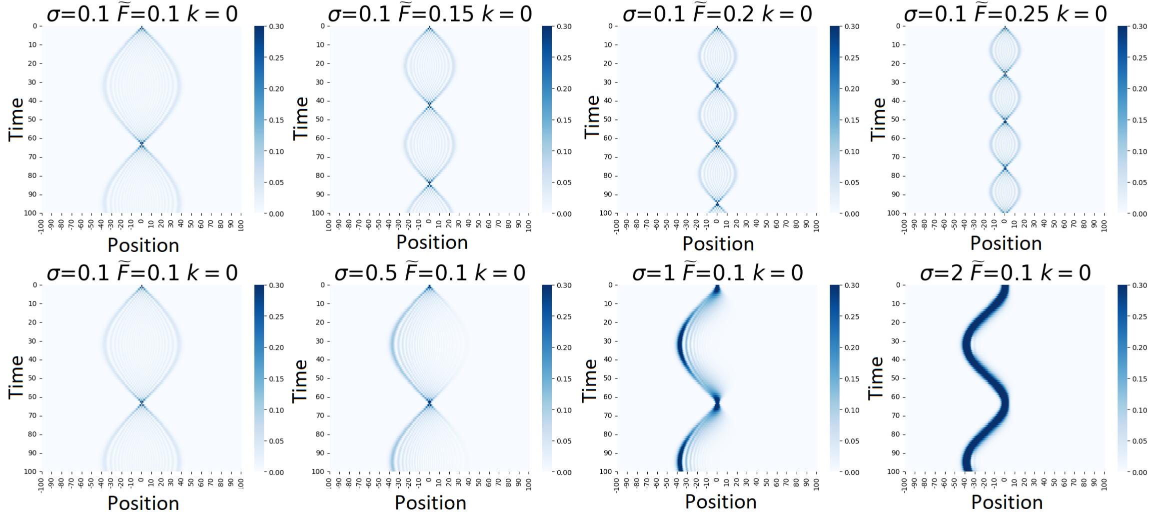
Center of a wavepacket oscillates and momentum k plays a role of a phase shift.

1D Case - wide wavepacket



Bloch oscillations

1D Case - narrow wavepacket



Bloch oscillations

Semiclassical approach

Bloch oscillations can be predicted by simple semiclassical analysis. The equation of motion of a single particle in linear potential Fx is given by

$$\frac{dp}{dt} = \hbar \frac{dk}{dt} = -F$$

which has the solution

$$k(t) = k(0) - \frac{Ft}{\hbar}.$$

Knowing dispersion relation for a free particle

$$E(k) = -2J \cos k,$$

and group velocity

$$v(k) = \frac{1}{\hbar} \frac{dE}{dk},$$

we obtain

$$x(t) = \int_0^t v(k(t'))dt' = \frac{2F}{J} \left[\cos\left(\frac{Ft}{\hbar} - k(0) \right) - \cos\left(k(0)\right) \right]$$

What are Lissajous figures?

A Lissajous figure is a trajectory of classical particle moving harmonically in two dimensions

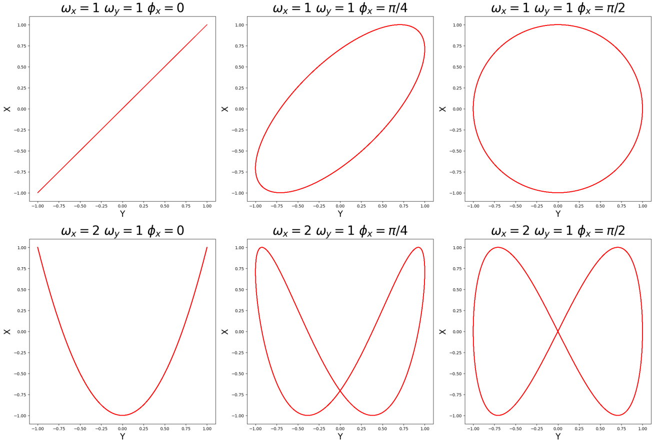
$$x(t) = A\cos(\omega_x t + \Delta\phi)$$

$$y(t) = B\cos\omega_y t$$

ω_x and ω_y are frequencies of oscillations in two perpendicular directions

Examples of Lissajous figures

$$\Delta\phi = 0 \quad \Delta\phi = \pi/4 \quad \Delta\phi = \pi/2$$



2D Case

Main problem of my work

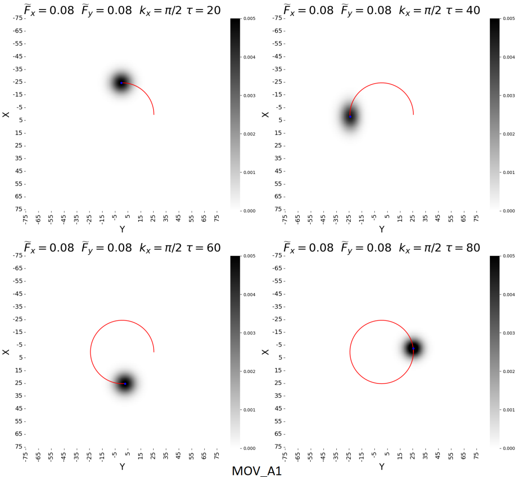
Is it possible to adjust forces in both directions in a way, that combined Bloch oscillations will result in trajectories analogous to the classical Lissajous figures?

In 2D case the Hamiltonian is given by:

$$\hat{H} = -J\sum_{xy} |x,y\rangle \left[\langle x+1,y| + \langle x-1,y| + \langle x,y+1| + \langle x,y-1|\right] + \sum_{xy} \left(\widetilde{F}_x x + \widetilde{F}_y y\right) |x,y\rangle \langle x,y|$$

It cannot be analytically diagonalized. Instead, we can solve it numerically

2D Case - circular trajectory



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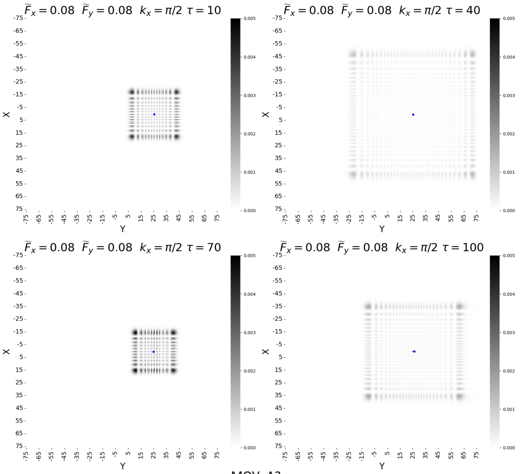
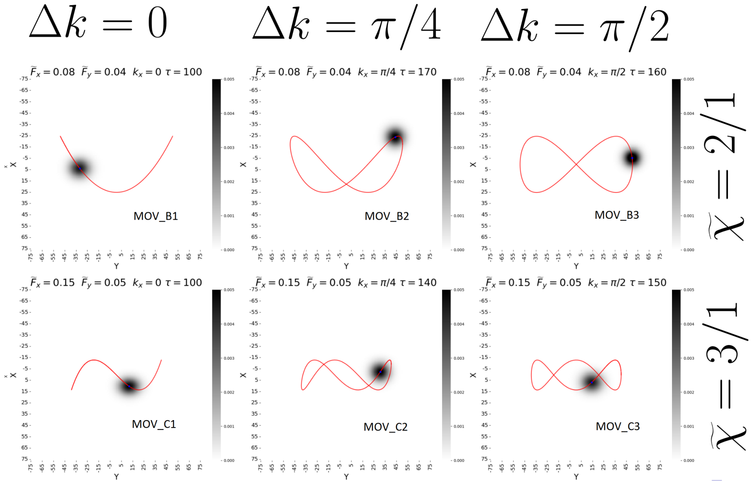


MOV_A3

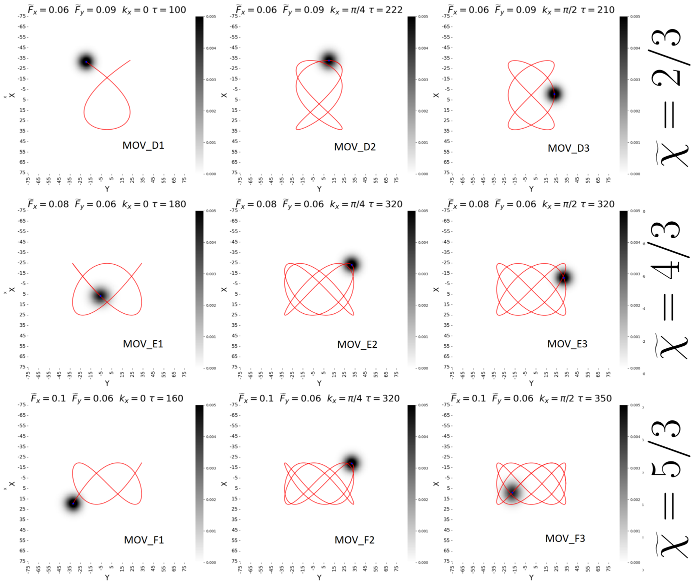
Circular trajectory can be obtained only for wavepackets that are wide enough.

2D Case - Lissajous figures

For wide wavepackets I obtained 15 differents Lissajous figures, besides circle. Their shapes depend on a ratio $\widetilde{\chi} = \widetilde{F}_x/\widetilde{F}_y$ and on difference of initial momenta $\Delta k = k_x - k_y$.



2D Case - Lissajous figures



Thank you for your attention