Functional representation of the S-matrix in Liouville theory

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Motivation

- Liouville Theory [Liouville 1853] is one of the most well-studied Conformal Field Theories
  - Scalar field in 1+1d in an exponential potential $V(\Phi) = 2\mu^2 e^{2\Phi}$, μ - "Cosmological constant"
  - Possesses conformal symmetry
  - Describes 2D gravity, bosonic strings [Dorn, Otto 1994] [Teschner 2001]

Setup

- $(\tau, \sigma) \in \mathbb{R}^1 \times \mathbb{S}^1$ - Cylinder

$$\partial_\tau^2 \Phi - \partial_\sigma^2 \Phi + 4\mu^2 e^{2\Phi} = 0$$

- Work in chiral coordinates

$$x = \tau + \sigma, \quad \bar{x} = \tau - \sigma$$

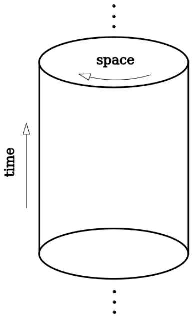
$$\Downarrow$$

- General solution

$$e^{-\Phi} = e^{-\Phi_{\text{in}}} + e^{-\Phi_{\text{out}}}$$

- Asymptotic fields

$$\Phi \xrightarrow{\tau \to \mp\infty} \Phi_{\text{in/out}}$$



Classical Solution

- $\Phi_{\text{in/out}}$ satisfy the free field equation (zero mode + oscillations)

$$\Phi_{\text{in}}(x,\bar{x}) = q + p\frac{x+\bar{x}}{2} - i\sum_{m>0}\frac{a_m}{m}e^{-imx} - i\sum_{m>0}\frac{\bar{a}_m}{m}e^{-im\bar{x}}$$

- Poisson brackets are given as

$$\{p,q\} = 1 \quad \{a_m, a_n\} = \frac{i}{2}m\delta_{m,-n} \quad \{\bar{a}_m, \bar{a}_n\} = \frac{i}{2}m\delta_{m,-n}$$

- For the out-field $q \to \tilde{q},\ p \to \tilde{p} = -p$ and $a_m \to b_m$

Classical solution

- Using the general solution and the Poisson brackets we can directly relate in and out fields [Teschner 2001]

$$e^{-\Phi_{\text{out}}(x,\bar{x})} = \mu^2 e^{-\Phi_{\text{in}}(x,\bar{x})} \frac{e^{2\pi p}}{\sinh^2(2\pi p)} \int_0^{2\pi} dy \int_0^{2\pi} d\bar{y}\, e^{2\Phi_{\text{in}}(x+y,\bar{x}+\bar{y})}$$

- Non-perturbative relation
- Aim:
  - Find a quantum version of this relation
  - Use it to study transition amplitudes

Quantization

- Canonical quantization[Dirac, 1982] maps modes to operators

$$p \to \hat{p},\ \ q \to \hat{q},\ \ a_n \to \hat{a}_n,\ \ a_{-n} \to \hat{a}_n^\dagger\ \ \ldots$$

and Poisson brackets to commutators

$$\{\cdot,\cdot\} \to \frac{i}{\hbar}[\cdot,\cdot]$$

Quantization

- Relation between quantum in/out fields [Teschner 2001]

$$: e^{-\Phi_{\text{out}}(x,\bar{x})} : = \frac{\mu_q^2}{2\sinh(\pi p)} \int_0^{2\pi} dy \int_0^{2\pi} d\bar{y} \; : e^{-\Phi_{\text{in}}(x,\bar{x})} :: e^{2\Phi_{\text{in}}(x+y,\bar{x}+\bar{y})-2\pi p} : \frac{1}{2\sinh(\pi p)}$$

- $\mu_q^2 = \mu^2 \frac{\sin(\pi\hbar)}{\pi\hbar}$ - Renormalized cosmological constant
- $: \cdots :$ - Normal ordering

Fock Space

- Asymptotic vacuum states - $|\cdot\rangle_{\text{in/out}}$ are defined as

$$_{\text{out}}\langle\tilde{p}|p\rangle_{\text{in}} = \delta(\tilde{p}+p)R(p)$$

- $R(p)$ is known as reflection amplitude and we can write

$$|p\rangle_{\text{in}} = R(p)|-p\rangle_{\text{out}}$$

Fock Space

- Asymptotic coherent states are defined using vacuum states

$$|p, a_k\rangle_{\text{in}} = \exp\left(\frac{2}{\hbar}\sum_{k>0}\frac{a_k}{k}\hat{a}_k^\dagger\right)|p\rangle_{\text{in}}$$

$$_{\text{out}}\langle\tilde{p}, b_l^*| = {}_{\text{out}}\langle\tilde{p}|\exp\left(\frac{2}{\hbar}\sum_{l>0}\frac{b_l^*}{l}\hat{b}_l\right)$$

- Coherent states are eigenstates of the creation operators

$$\hat{a}_k|p, a_k\rangle_{\text{in}} = a_k|p, a_k\rangle_{\text{in}}$$

$$_{\text{out}}\langle\tilde{p}, b_l^*|b_l^\dagger = {}_{\text{out}}\langle\tilde{p}, b_l^*|b_l^*$$

S-Matrix

- Definition of the S-Matrix as the transition amplitude

$$_{\text{out}}\langle \tilde{p}, b_l^* | p, a_k \rangle_{\text{in}} = \delta(\tilde{p} + p) R(p) S_p(b_l^*, a_k)$$

- *Note:*

$$\exp\left(\frac{2}{\hbar}\sum_{l>0}\frac{b_l^*}{l}\hat{b}_l\right)|p\rangle_{\text{in}} = |p\rangle_{\text{in}} \quad \Rightarrow \quad S_p(b_l^*, 0) = 1$$

and similarly $S_p(0, a_k) = 1$

Solution

- Aim: Find expressions for $S_p(b_l^*, a_k)$ and $R(p)$
- By projecting the asymptotic field relation between the coherent states we derive a recurrence in a_k and b_l^*
- By setting $a_k = b_l^* = 0$ we get function equation for $R(p)$

$$R(p) = \frac{\pi^2 \mu_q^2 \gamma_p^2}{\sinh(\pi p) \sinh(\pi p - i\pi\hbar)} R(p - i\hbar)$$

with

$$\gamma_p = \int_0^{2\pi} \frac{dy}{2\pi} e^{(p-i\hbar)(y-\pi)} (1 - e^{iy})^{\hbar} = \frac{\Gamma(1+\hbar)}{\Gamma(1-ip)\Gamma(1+\hbar+ip)}$$

Reflection Amplitude

- This recurrence is solved to get

$$R(p) = -(\mu_q^2 \Gamma^2(\hbar))^{-\frac{ip}{\hbar}} \frac{\Gamma(ip/\hbar)\Gamma(ip)}{\Gamma(-ip/\hbar)\Gamma(-ip)}$$

- The result is in excellent agreement with the well known DOZZ formula
  [Dorn, Otto 1994]
  [Zamolodchikov, Zamolodchikov 1996]

First Level Transition

- Using the expression for $R(p)$ recursion for the S-matrix is derived
- In the case of $b_l^* = 0$ it simplifies to

$$S_p\left(-\frac{i\hbar}{2}e^{-ilx}, a_k\right) = \exp\left(-i\sum_{m>0}\frac{a_m}{m}e^{-imx}\right)\times$$

$$\frac{1}{\gamma_p}\int_0^{2\pi}\frac{dy}{2\pi}e^{(p-i\hbar)(y-\pi)}(1-e^{iy})^{\hbar}\exp\left(2i\sum_{m>0}\frac{a_m}{m}e^{-im(x+y)}\right)$$

First Level Transition

- S is holomorphic $\Rightarrow$ Expand and collect terms linear in a_1 to get the First level transition

$$S_{-1,1}(p) = \frac{2}{\hbar} \frac{1 + \hbar - ip}{1 + \hbar + ip}$$

- Similar to $R(p)$ this also agrees with the results in [Zamolodchikov, Zamolodchikov 1996]

Solution

- To continue we define

$$U_p(b_l^*, a_k) = \exp\left(-\frac{2}{\hbar}\sum_{m>0}\frac{b_m^* a_m}{m}\right) S_p(b_l^*, a_k)$$

- At $b_l^* = a_l^*$ introduce new notation $U_p(a_n)$ which coincides with the normal symbol for the S-matrix with $n \neq 0$

Solution

- At $p = i\hbar N$ where $N \in \mathbb{Z}^+$ we can write a recursive relation

$$U_{i\hbar N}(a_n) = \frac{1}{\gamma_{i\hbar N}} \int_0^{2\pi} \frac{dy}{2\pi} e^{i(N-1)\hbar(y-\pi)} \exp\left(2i \sum_{m>0} \frac{a_m}{m} e^{-im(x+y)}\right) \times U_{i\hbar(N-1)}\left(a_n + \frac{i\hbar}{2} e^{inx} - i\hbar\theta_n e^{in(x+y)}\right)$$

- With

$$\theta_n = \begin{cases} 0 & n < 0 \\ 1 & n > 0 \end{cases}$$

Solution

- Assuming no scattering at $p = 0$ i.e. $U_0(a_n) = 1$ we get the same result as from the path integral representation of the S-matrix [Jorjadze, Theisen 2021]

$$U_{i\hbar}(a_n) = 1 + \sum_{\nu\geq 2} \frac{1}{\nu!} \sum_{n_1\cdots n_\nu} \frac{j_{n_1}\cdots j_{n_\nu}}{|n_1|\cdots|n_\nu|}\delta_{n_1\cdots n_\nu}, \quad j_n = 2i\epsilon(n)a_n$$

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Solution

- We can analytically write the solution of $U_{i\hbar N}(a_n)$ in the form of an N-dimensional integral

$$U_N(a_n) = \prod_{\alpha=1}^{N} \frac{1}{\gamma_{i\hbar\alpha}} \oint \frac{d\zeta_1}{2\pi i\zeta_1} \cdots \oint \frac{d\zeta_N}{2\pi i\zeta_N}$$

$$\exp\left(2i\sum_{n>0} \frac{a_n}{n}(\zeta_1^{-n} + \cdots \zeta_N^{-n})\right) \prod_{1\leq\alpha<\beta\leq N} (\zeta_\alpha\zeta_\beta)^{\hbar}(\zeta_\alpha - \zeta_\beta)^{-2\hbar}$$

- These are Dotsenko-Fateev - type integrals
- The results can be analytically continued to $N \notin \mathbb{Z}^+$

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Thank You!

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Quantum Fields

$$: e^{-\Phi_{\mathrm{out}}(x,\bar{x})} : = e^{-q_{\mathrm{out}}+p\tau}\,\bar{E}^{(-1)\dagger}_{\mathrm{out}}(\bar{x})E^{(-1)\dagger}_{\mathrm{out}}(x)E^{(-1)}_{\mathrm{out}}(x)\bar{E}^{(-1)}_{\mathrm{out}}(\bar{x})$$

$$: e^{-\Phi_{\mathrm{in}}(x,\bar{x})} : = e^{-q_{\mathrm{in}}+p\tau}\,\bar{E}^{(-1)\dagger}_{\mathrm{in}}(\bar{x},)E^{(-1)\dagger}_{\mathrm{in}}(x)E^{(-1)}_{\mathrm{in}}(x)\bar{E}^{(-1)}_{\mathrm{in}}(\bar{x})$$

$$: e^{2\Phi_{\mathrm{in}}(z,\bar{z})} : = e^{2q_{\mathrm{in}}+p(z+\bar{z}-2\pi)}\,\bar{E}^{(2)\dagger}_{\mathrm{in}}(\bar{z})E^{(2)\dagger}_{\mathrm{in}}(z)E^{(2)}_{\mathrm{in}}(z)\bar{E}^{(2)}_{\mathrm{in}}(\bar{z})$$

$$E^{(-1)}_{\mathrm{in}}(x) = \exp\left(-i\sum_{m>0}\frac{a_m}{m}e^{-imx}\right)$$

$$E^{(2)}_{\mathrm{in}}(z) = \exp\left(2i\sum_{m>0}\frac{a_m}{m}e^{-imz}\right)$$

Quantum Fields

$$
{}_{\text{out}}\langle \tilde{p}, b_l^* | : e^{-\Phi_{\text{out}}(x,\bar{x})} : | p, a_k \rangle_{\text{in}} = \delta(\tilde{p} + p - i\hbar) e^{(p - i\hbar/2)\tau} R(p)
$$

$$
\exp\left(i \sum_{m>0} \frac{b_m^*}{m} e^{imx} \right) S_p \left(b_m^* - \frac{i\hbar}{2} e^{-ilx}, a_k \right)
$$

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