Geometric Quantum Mechanics: Greening Quantum Gravity's Arid Boxes

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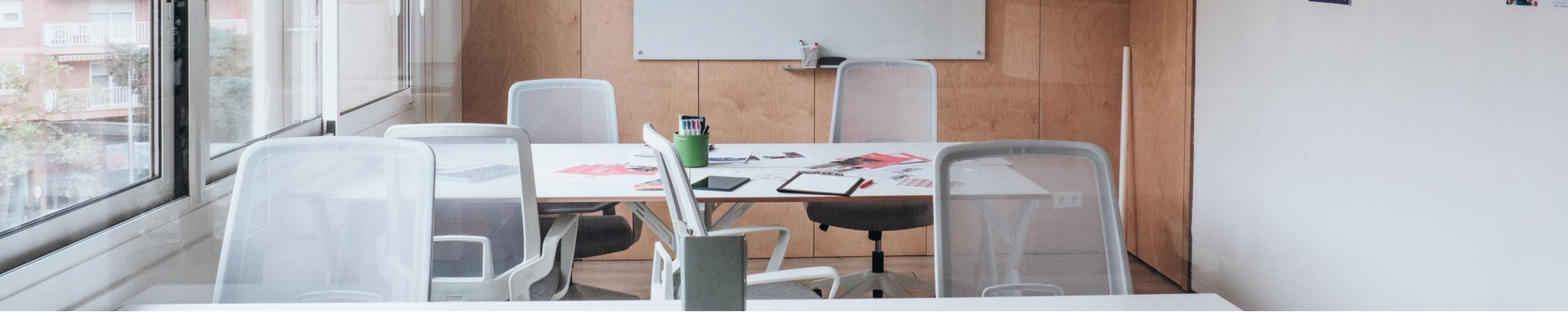

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Greetings!

Warm greetings to all present. Based on arXiv:2201.01187v1 [quant-ph] in collaboration with Dr. Akrami, Mathematics Department, IPM Institute in Fundamental Sciences, Tehran, Iran.

Overview

“Scene 1”

Prologue:
A Query In Quantum Formulation

A Query In Quantum Formulation

Exploiting general Theory of Relativity into Quantum Theory, or vise versa?

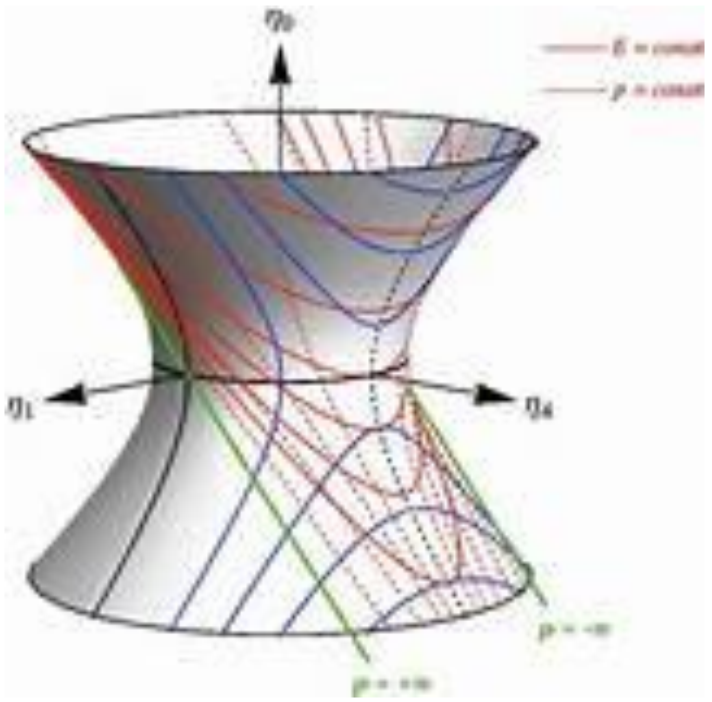


Figure I. Roughly picturing Minkowskian Space

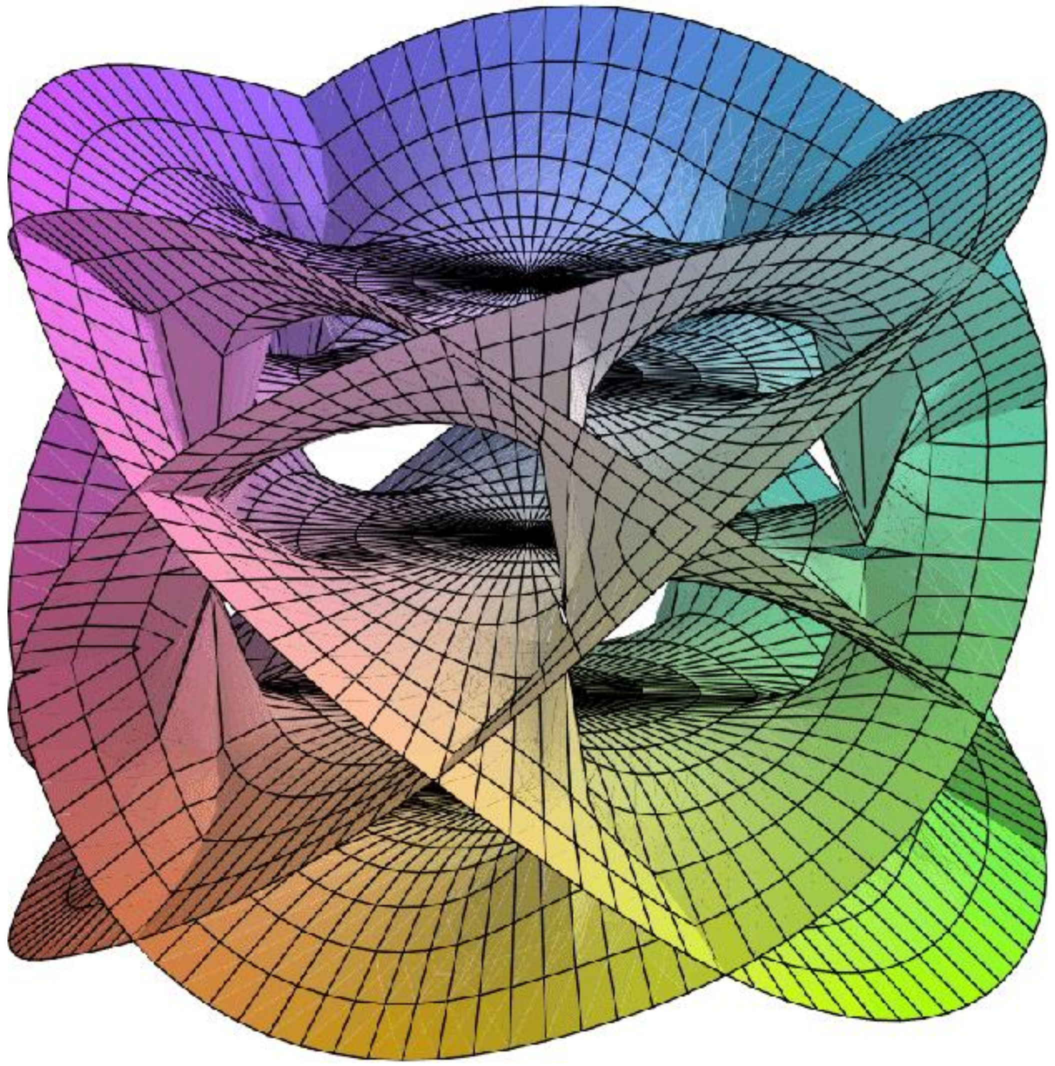
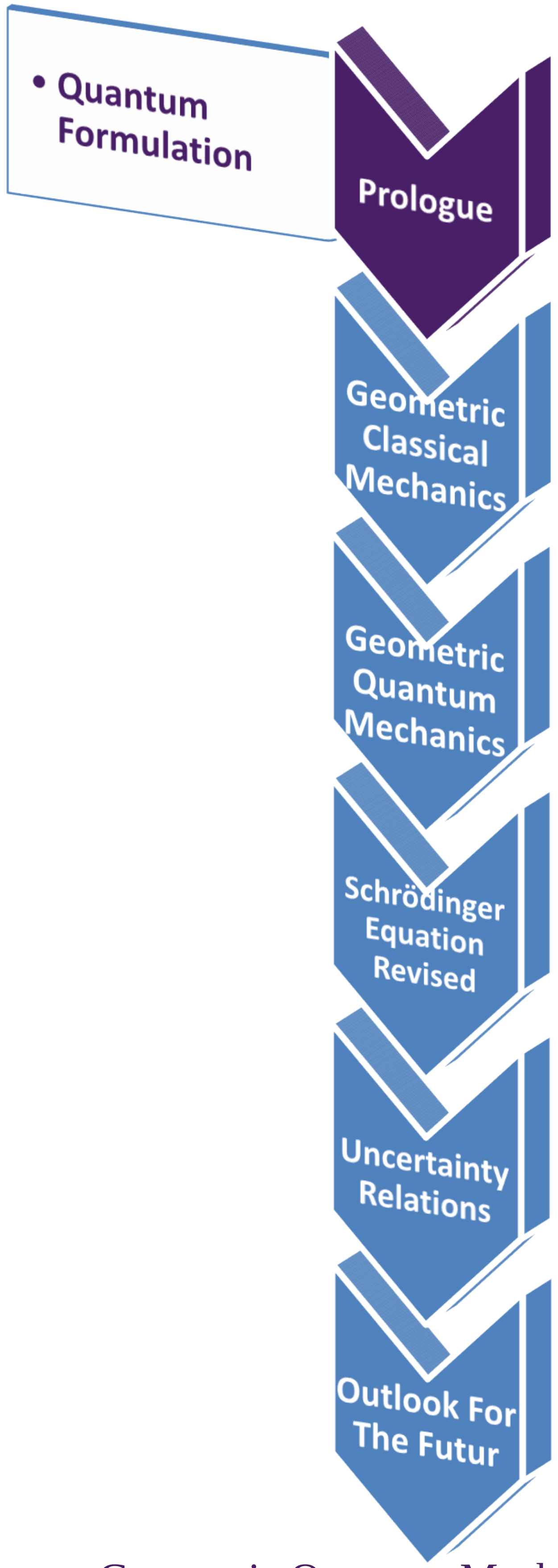


Figure II. Roughly picturing Riemannian Manifold (Γ, g)



“Scene II”

Geometric Classical Mechanics: On The Virtue of Manifolds

Differential and Manifold Geometry

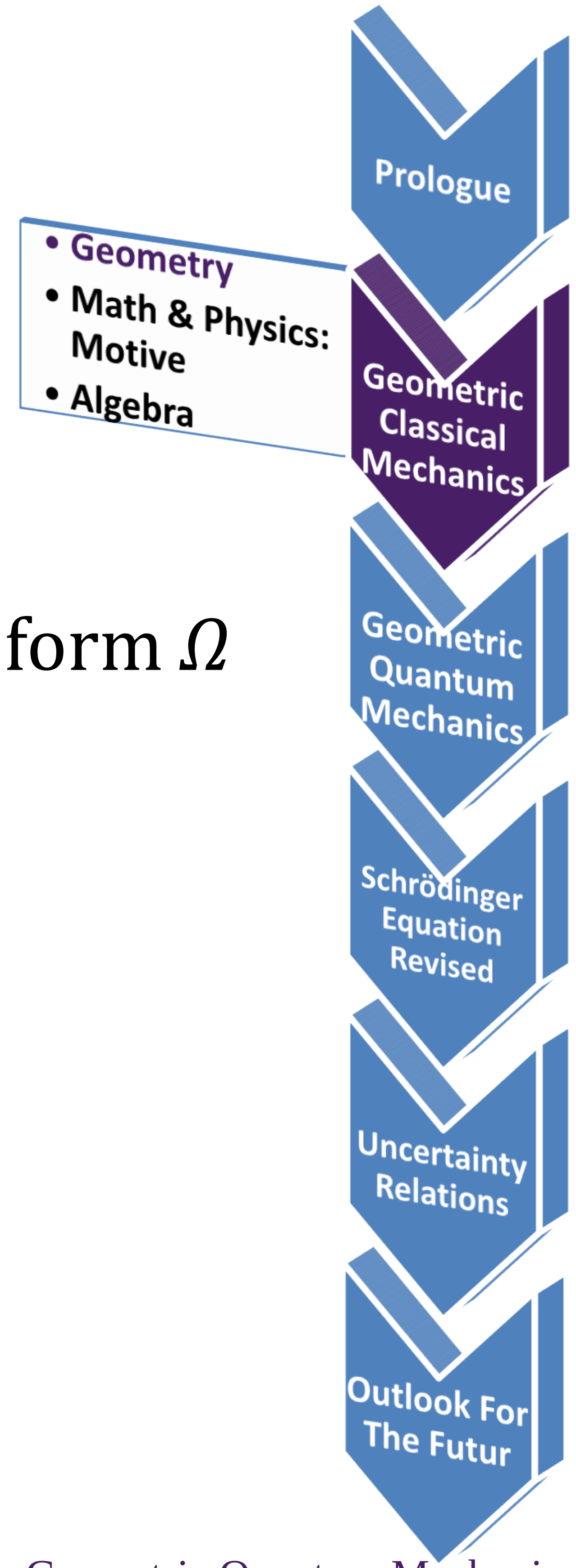
Mathematical Frame: Symplectic Manifold

i. Ω_{ab} : symplectic form, i.e. a closed non-degenerate 2-form

ii. (Γ, Ω): symplectic manifold, i.e. a manifold Γ equipped with the symplectic form Ω

iii. $Tp(\Gamma) \cong \left(O_{\Gamma,p} \,/\, S_{\Gamma,p}\right)^{*}$: Tangent space for $p \in \Gamma$

Observation.
Phase Space is a Symplectic Manifold!



Mathematical Motive

Structure – Preserving Diffeomorphisms?

i. Vector field $X: \Gamma \to T(\Gamma)$, symplectic structure preserved under "motion" of X along Γ if $d i_X \Omega = 0$

ii. $i_X \Omega$ is closed and exact

$$\Rightarrow \exists f, f: \Gamma \to \mathbb{R}\ s.t.\ i_x \Omega = df$$

$$\Rightarrow {X_f}^a = \Omega^{ab} \partial_a f^a$$

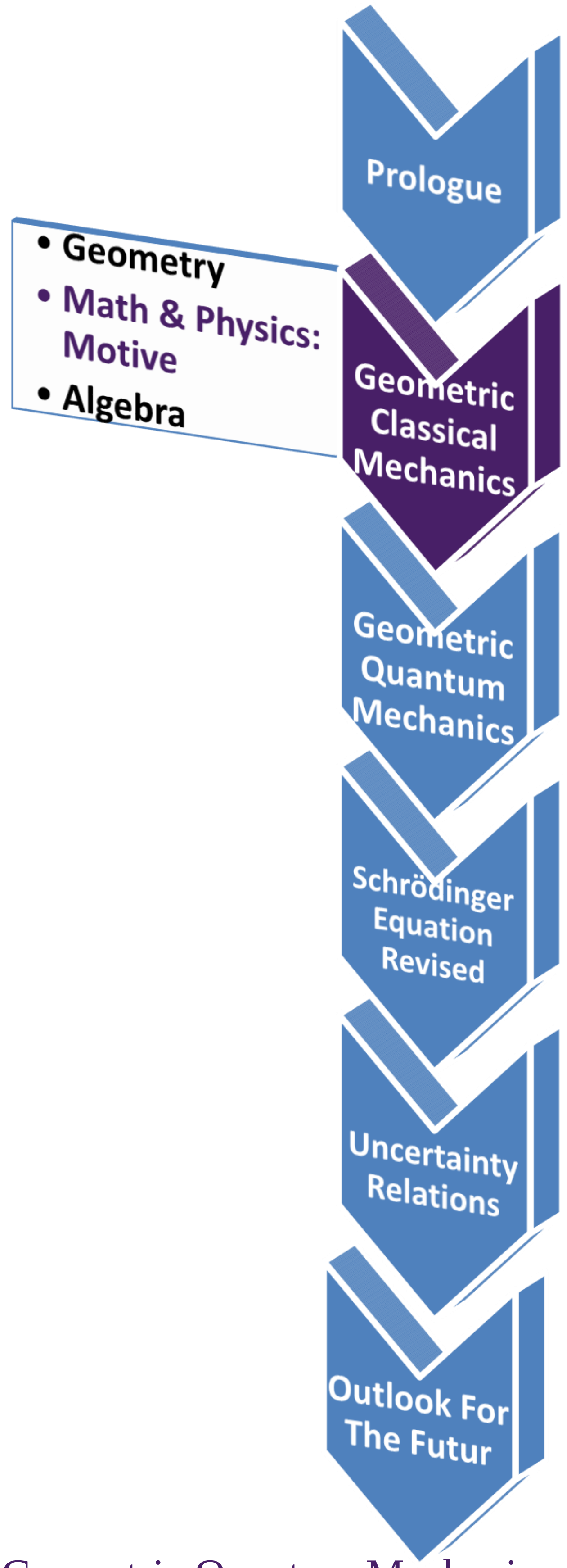
Physical Motive

Evolution of Physical System?

i. f is an observable

ii. X is a locally Hamiltonian Vector Field

Intro to Hamiltonian Mechanics:
Canonical transformations!



Algebra

Topological Space of Observables

$$O_{cl} \coloneqq \{f \,|\, f : \Gamma \to \mathbb{R}, smooth\}$$

Poisson Bracket

$$For\ F, G\ \epsilon O_{cl}, \qquad \{\hat{F}, \hat{G}\}_{cl}$$

$$\coloneqq (\partial_a f)\Omega^{db}(\partial_b g) = \Omega(X_F, X_G)$$

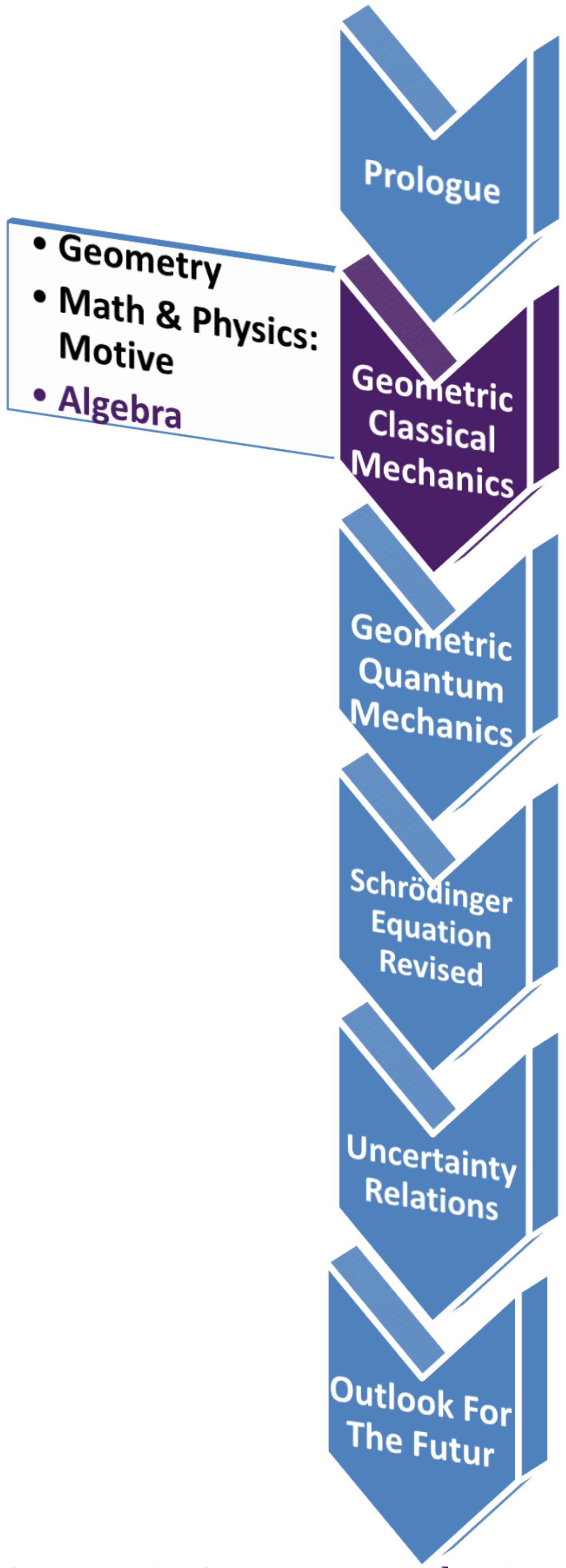
$\Rightarrow$ A Lie "structure" on O_{qu} !

Evolution

Derivation

$$\mathcal{L}_{X_f}\Omega = 0 \Rightarrow \mathcal{L}_X i_Y - i_Y \mathcal{L}_X$$
$$= i_{[X,Y]}$$

Therefore,
Hamilton's
equations in hand!



“Scene III ”

Geometric Quantum Theory:
Geometry mirrors Algebra

Quantum Algebra

Algebraic Structures on O_{qu}

i. Lie Bracket

$$For\ \hat{F}, \hat{G} \epsilon O_{qu}, \qquad \{\hat{F}, \hat{G}\}_{qu} := \frac{1}{i\hbar}[\hat{F}, \hat{G}]$$

$\Rightarrow$ A Lie "structure" on O_{qu} !

ii. Jordan Product

$$\{\hat{F}, \hat{G}\}_{+} := \frac{1}{2}[\hat{F}, \hat{G}]_{+}$$

$\Rightarrow$ A commutative structure on O_{qu}!

Classical Analogue

i. Dirac Quantization

$$\begin{cases} \widehat{\{f, g\}_{cl}} = [\hat{F}, \hat{G}]_{qu} \\ \frac{1}{2}[\hat{F}, \hat{G}]_{+} = fg \text{ (pointwise)} \end{cases}$$

ii. Derivation

For $\hat{F}, \hat{G}, \hat{H} \epsilon O_{qu}$,

$$\left\{\hat{F}, \{\hat{G}, \hat{H}\}_{+}\right\}_{+} = \left\{\hat{G}, \{\hat{F}, \hat{H}\}_{qu}\right\}_{+} + \left\{\{\hat{F}, \hat{G}\}_{qu}, \hat{H}\right\}_{+}$$

PROFOUND SIMILARITIES, YET A CRUCIAL DIFFERENCE: Non-associativity of Jordan Product

However, is under control!

$$\left\{\hat{F}, \{\hat{G}, \hat{H}\}_{+}\right\}_{+} - \left\{\{\hat{F}, \hat{G}\}_{+}, \hat{H}\right\}_{+} = \left(\frac{\hbar}{2}\right)^2 \left\{\hat{G}, \{\hat{F}, \hat{H}\}_{qu}\right\}_{qu}$$

Geometrizing Algebraic Structure

Expressing Algebraic Properties On H

$\boldsymbol{H}$ as a real vector space. Then:

i. Complex Structure On Real Vector Space

$$J: H \to H$$
$$\Psi \mapsto i\Psi$$

ii. Inner Product On Real Vector Space

For $\Phi, \Psi \epsilon H$, $\quad \langle \Phi, \Psi \rangle = \frac{1}{2\hbar} G(\Phi, \Psi) + \frac{i}{2\hbar} \Omega(\Phi, \Psi)$

$$\langle \Phi, \Psi \rangle = \overline{\langle \Phi, \Psi \rangle} \Rightarrow \begin{cases} \langle \Phi, \Phi \rangle = \langle J\Phi, J\Phi \rangle & (1.1) \\ G(\Phi, \Psi) = G(\Psi, \Phi) & (1.2) \\ \Omega(\Phi, \Psi) = -\Omega(\Psi, \Phi) & (1.3) \end{cases}$$

Relationship Between G, Ω and J

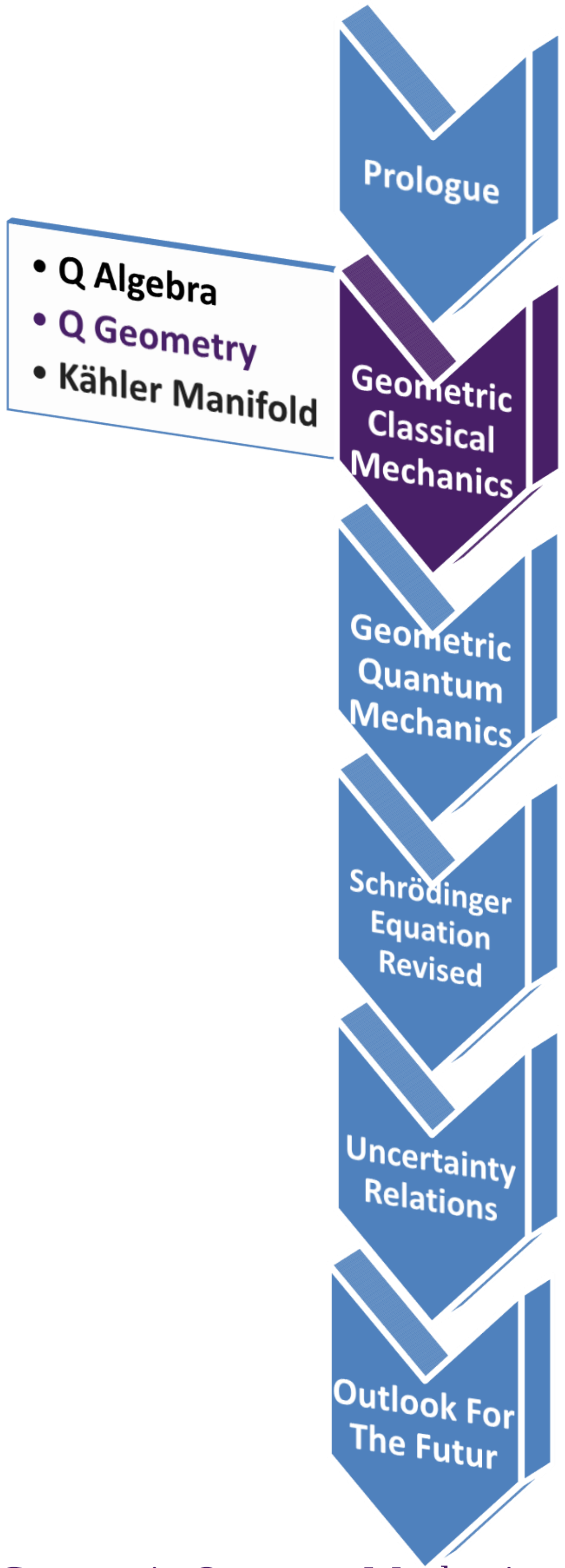
$$G(J\Phi, J\Psi) = G(\Psi, \Phi) \text{ and } \Omega(J\Phi, J\Psi) = \Omega(\Psi, \Phi) \quad (2)$$

Exploiting The Properties In Differential Topology

As a symplectic manifold:

$$\Rightarrow \begin{cases} J \text{ is an isometric symplectomorphism by (1.1), (2)} \\ G \text{ isa positive} - \text{definite real inner product by (1.2)} \\ \Omega \text{ is skew} - \text{symmetric by (1.3)} \end{cases}$$

In particular, as a nice Kähler manifold!



H As Kähler Manifold

Construction of Kähler manifold

Through the lens of Differential Geometry:

$$\Omega, G: T_\psi H \times T_\psi H \to \mathbb{R}$$

strongly non-degenerate bilinear forms

The Final Frame:

J: complex structure

G: real inner product

Ω: skew-symmetric closed 2-form ($d_\psi \Omega = 0$)

$\Rightarrow (H, \Omega, J)$ is a Kähler manifold equipped with real inner-product G!

“Scene ɪᴠ”

Schrödinger Equation: Mathematical

Vector Fields on H

Observation.

$$\hat{F} \in O_{qu}\,;\ \hat{F}: H \to H, \Psi \mapsto \hat{A}\Psi$$

$$\Rightarrow is\ by\ definition\ a\ vector\ field\ on\ H$$

Associated Schrödinger vector field:

$$Y_{\hat{F}}: H \to H, \Psi \mapsto -\frac{1}{\hbar} J\hat{F}\Psi$$

Derivation of Schrödinger equation:

$$(dF)(\eta) = \frac{d}{dt}\langle \Psi + t\eta, \hat{F}(\Psi + t\eta)\rangle\Big|_{t=0} = \langle \Psi, \hat{F}\eta\rangle + \langle \eta, \hat{F}\Psi\rangle = 2\,\mathrm{Re}\langle \eta, \hat{F}\psi\rangle$$

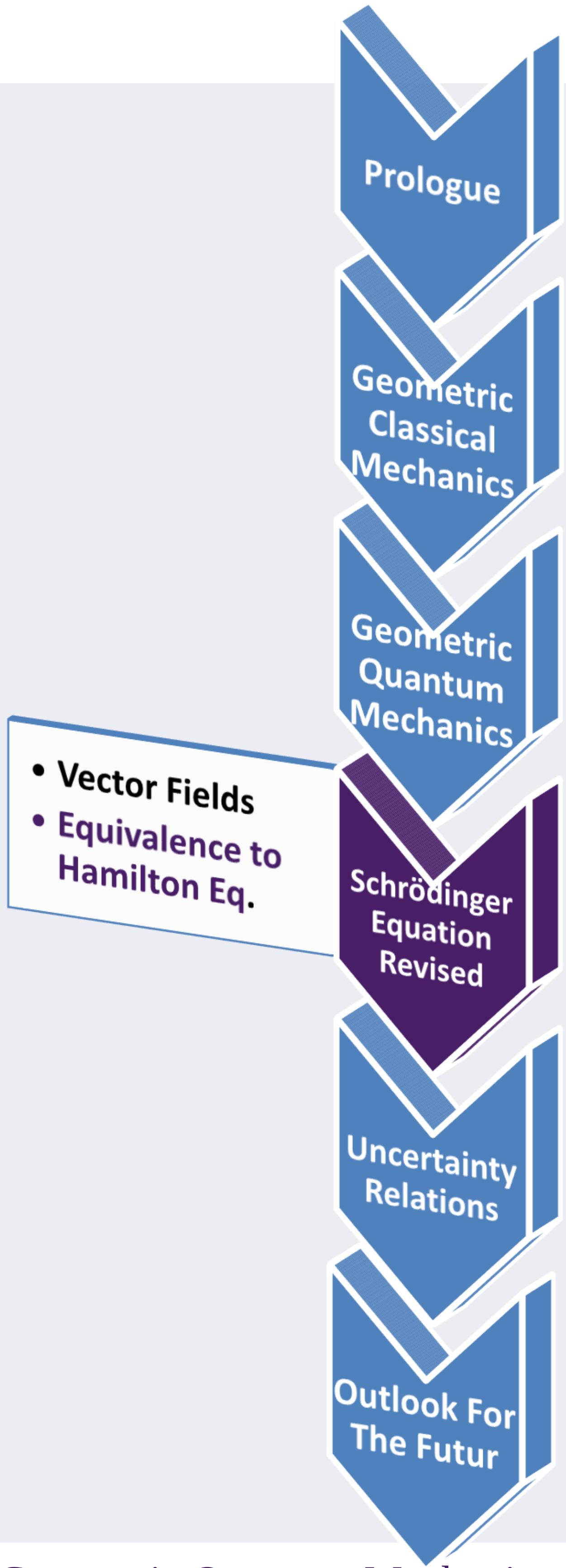
$$= \frac{1}{\hbar} G\left(\hat{F}\Psi, \eta\right) = G(JY_{\hat{F}}(\Psi), \eta) = \Omega(Y_{\hat{F}}, \eta) = i_{Y_F}\,\Omega$$

Equivalence to Hamilton's Equation

Theorem.

The Schrödinger vector field $Y_{\hat{F}}$ *determined by the observable* $\hat{F} \in O_{qu}$

$\Leftrightarrow$ *Hamiltonian vector field* X_F *generated by the expectation value of* F.



“Scene vi”

Uncertainty Relations:
Back To Algebra

i. Poisson Brackets

Algebraic Operations On The Expectation value Functions

Induced by the Lie bracket:

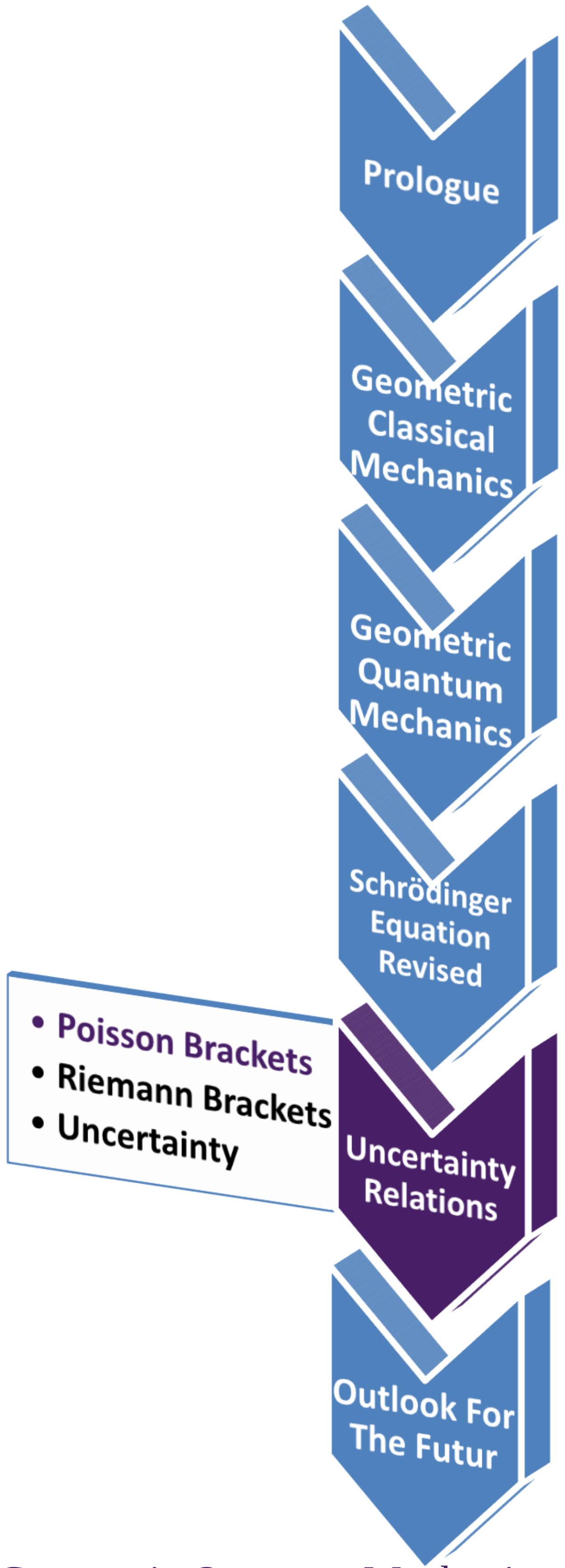
$$For\ \hat{F}, \hat{K} \in O_{qu}, \qquad \left\langle \{\hat{F}, \hat{K}\}_{qu} \right\rangle : H \to \mathbb{R}$$

$$\left\langle -\frac{1}{\hbar} J[\hat{F}, \hat{K}] \right\rangle (\Psi) = \frac{1}{i\hbar}\left(\langle \hat{F}\Psi, \hat{K}\Psi\rangle - \langle \hat{K}\Psi, \hat{F}\Psi\rangle\right) = \frac{2}{\hbar} \mathrm{Im}\langle \hat{F}\psi, \hat{K}\psi\rangle = \frac{1}{\hbar^2}\Omega\left(\hat{F}\psi, \hat{K}\psi\right)$$

$$\Rightarrow \langle Y_{[\hat{F},\hat{K}]}\rangle(\Psi) = \Omega(Y_{\hat{F}}, Y_{\hat{K}}) = \Omega(X_{\hat{F}}, X_{\hat{K}}) = \{F, K\}$$

$$\Rightarrow The\ induced\ algebraic\ operation\ is\ a\ Poisson\ Bracket.$$

(With respect to the quantum symplectic structure!)



ii. *Riemann Brackets*

Algebraic Operations On The Expectation value Functions

Induced by the Jordan product:

$$\left\langle \{\hat{F},\hat{K}\}_+ \right\rangle : H \to \mathbb{R}$$

$$\left\langle \frac{1}{2}[\hat{F},\hat{K}]_+ \right\rangle(\Psi) = \frac{1}{2}\left(\langle \hat{F}\Psi,\hat{K}\Psi\rangle + \langle \hat{K}\Psi,\hat{F}\Psi\rangle\right) = \frac{1}{2\hbar}G\left(\hat{F}\psi,\hat{K}\psi\right)$$

$$\Rightarrow \left\langle \frac{1}{2}[\hat{F},\hat{K}]_+ \right\rangle(\Psi) = \frac{\hbar}{2}G(Y_{\hat{F}},Y_{\hat{K}}) = \frac{\hbar}{2}G(X_{\hat{F}},X_{\hat{K}})$$

$\Rightarrow$ *The induced algebraic operation is a Riemann Bracket.*

Uncertainty of Observables

Algebraic Operations On The Expectation value Functions

Induced by the Lie bracket:

$$(\Delta\hat{F})^2 = \langle \hat{F} \rangle - F^2 = \{\hat{F}_1 \hat{F}\}_+ - F^2,$$

$$\left\langle -\frac{1}{\hbar} J[\hat{F},\hat{K}] \right\rangle(\Psi) = (\Delta\hat{F})^2 (\Delta\hat{K})^2 \geq \left\langle -\frac{i}{2}[\hat{F},\hat{K}] \right\rangle^2 = \left(\frac{\hbar}{2}\{F,K\}_{qu} \right)$$

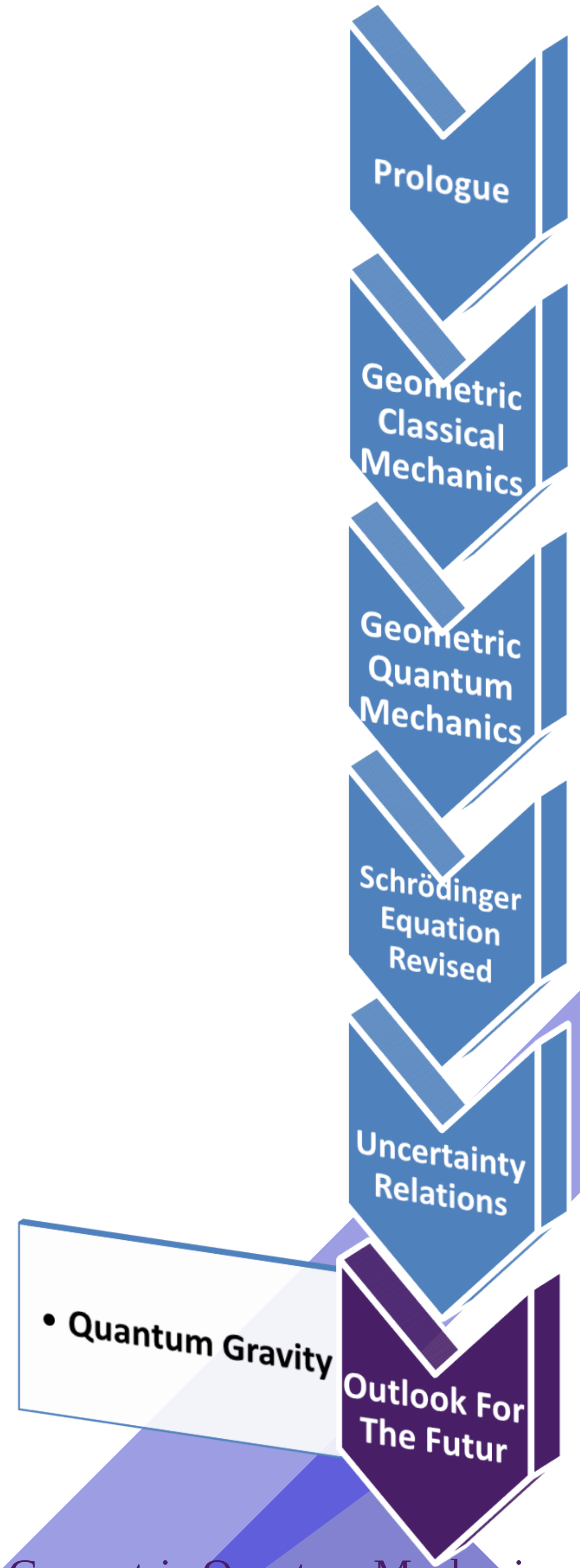
$$\Rightarrow (\Delta\hat{F})^2 (\Delta\hat{K})^2 \geq \left\langle -\frac{i}{2}[\hat{F},\hat{K}] \right\rangle^2 + \left\langle \frac{1}{2}\left[\widehat{F'},\widehat{K'}\right]_+ \right\rangle^2, \widehat{F'}(\Psi) := \hat{F}(\Psi) - F(\Psi)$$

$$\Rightarrow (\Delta\hat{F})^2 \, (\Delta\hat{K})^2 \geq \left(\frac{\hbar}{2}\{F,K\}_{qu} \right)^2 + (\{F,K\}_+ - FK)^2$$

(With respect to the quantum symplectic structure!)

Conclusion & Outlook For The Futur

1. Exploiting more algebraic properties into geometric properties
2. Proposing Killing vector fields on the Hilbert space
3. Proposing Killing vector fields on the Hilbert space
4. Imbedding gravity in these geometric properties



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[Curtain.]

Thank you!

10 Aug, 2024