Finding Symmetries of $y''' = yy'' - (y')^2$

```
eta1 = D[η[x, y[x]], x] - y'[x] × D[ξ[x, y[x]], x]
```

$$y'[x]\,\eta^{(0,1)}[x, y[x]] + \eta^{(1,0)}[x, y[x]] - y'[x]\left(y'[x]\,\xi^{(0,1)}[x, y[x]] + \xi^{(1,0)}[x, y[x]]\right)$$

```
eta2 = D[eta1, x] - y''[x] × D[ξ[x, y[x]], x]
```

$$\begin{aligned}&y''[x]\,\eta^{(0,1)}[x, y[x]] - 2\,y''[x]\left(y'[x]\,\xi^{(0,1)}[x, y[x]] + \xi^{(1,0)}[x, y[x]]\right) + \\ &\ y'[x]\,\eta^{(1,1)}[x, y[x]] + y'[x]\left(y'[x]\,\eta^{(0,2)}[x, y[x]] + \eta^{(1,1)}[x, y[x]]\right) + \\ &\ \eta^{(2,0)}[x, y[x]] - y'[x]\left(y''[x]\,\xi^{(0,1)}[x, y[x]] + y'[x]\,\xi^{(1,1)}[x, y[x]] + \right. \\ &\qquad \left. y'[x]\left(y'[x]\,\xi^{(0,2)}[x, y[x]] + \xi^{(1,1)}[x, y[x]]\right) + \xi^{(2,0)}[x, y[x]]\right)\end{aligned}$$

```
eta3 = D[eta2, x] - y'''[x] × D[ξ[x, y[x]], x]
```

$$\begin{aligned}&y^{(3)}[x]\,\eta^{(0,1)}[x, y[x]] - 3\,y^{(3)}[x]\left(y'[x]\,\xi^{(0,1)}[x, y[x]] + \xi^{(1,0)}[x, y[x]]\right) + \\ &\ y''[x]\,\eta^{(1,1)}[x, y[x]] + 2\,y''[x]\left(y'[x]\,\eta^{(0,2)}[x, y[x]] + \eta^{(1,1)}[x, y[x]]\right) - \\ &\ 3\,y''[x]\left(y''[x]\,\xi^{(0,1)}[x, y[x]] + y'[x]\,\xi^{(1,1)}[x, y[x]] + \right. \\ &\qquad \left. y'[x]\left(y'[x]\,\xi^{(0,2)}[x, y[x]] + \xi^{(1,1)}[x, y[x]]\right) + \xi^{(2,0)}[x, y[x]]\right) + \\ &\ y'[x]\,\eta^{(2,1)}[x, y[x]] + y'[x]\left(y'[x]\,\eta^{(1,2)}[x, y[x]] + \eta^{(2,1)}[x, y[x]]\right) + \\ &\ y'[x]\left(y''[x]\,\eta^{(0,2)}[x, y[x]] + y'[x]\,\eta^{(1,2)}[x, y[x]] + \right. \\ &\qquad \left. y'[x]\left(y'[x]\,\eta^{(0,3)}[x, y[x]] + \eta^{(1,2)}[x, y[x]]\right) + \eta^{(2,1)}[x, y[x]]\right) + \\ &\ \eta^{(3,0)}[x, y[x]] - y'[x]\left(y^{(3)}[x]\,\xi^{(0,1)}[x, y[x]] + y''[x]\,\xi^{(1,1)}[x, y[x]] + \right. \\ &\qquad 2\,y''[x]\left(y'[x]\,\xi^{(0,2)}[x, y[x]] + \xi^{(1,1)}[x, y[x]]\right) + \\ &\qquad y'[x]\,\xi^{(2,1)}[x, y[x]] + y'[x]\left(y'[x]\,\xi^{(1,2)}[x, y[x]] + \xi^{(2,1)}[x, y[x]]\right) + \\ &\qquad y'[x]\left(y''[x]\,\xi^{(0,2)}[x, y[x]] + y'[x]\,\xi^{(1,2)}[x, y[x]] + \right. \\ &\qquad\qquad \left. \left. y'[x]\left(y'[x]\,\xi^{(0,3)}[x, y[x]] + \xi^{(1,2)}[x, y[x]]\right) + \xi^{(2,1)}[x, y[x]]\right) + \xi^{(3,0)}[x, y[x]]\right)\end{aligned}$$

```
rrule = {y[x] → Y0, y'[x] → Y1, y''[x] → Y2, y'''[x] → Y3};
```

Replace[eta1, rrule] and eta1 /. rrule are same

```
Eta1 = eta1 /. rrule
Eta2 = eta2 /. rrule
Eta3 = eta3 /. rrule
```

$$Y1\,\eta^{(0,1)}[x, Y0] + \eta^{(1,0)}[x, Y0] - Y1\left(Y1\,\xi^{(0,1)}[x, Y0] + \xi^{(1,0)}[x, Y0]\right)$$

$$\begin{aligned}&Y2\,\eta^{(0,1)}[x, Y0] - 2\,Y2\left(Y1\,\xi^{(0,1)}[x, Y0] + \xi^{(1,0)}[x, Y0]\right) + \\ &\quad Y1\,\eta^{(1,1)}[x, Y0] + Y1\left(Y1\,\eta^{(0,2)}[x, Y0] + \eta^{(1,1)}[x, Y0]\right) + \eta^{(2,0)}[x, Y0] - \\ &\quad Y1\left(Y2\,\xi^{(0,1)}[x, Y0] + Y1\,\xi^{(1,1)}[x, Y0] + Y1\left(Y1\,\xi^{(0,2)}[x, Y0] + \xi^{(1,1)}[x, Y0]\right) + \xi^{(2,0)}[x, Y0]\right)\end{aligned}$$

$$\begin{aligned}&Y3\,\eta^{(0,1)}[x, Y0] - 3\,Y3\left(Y1\,\xi^{(0,1)}[x, Y0] + \xi^{(1,0)}[x, Y0]\right) + \\ &\quad Y2\,\eta^{(1,1)}[x, Y0] + 2\,Y2\left(Y1\,\eta^{(0,2)}[x, Y0] + \eta^{(1,1)}[x, Y0]\right) - \\ &\quad 3\,Y2\left(Y2\,\xi^{(0,1)}[x, Y0] + Y1\,\xi^{(1,1)}[x, Y0] + Y1\left(Y1\,\xi^{(0,2)}[x, Y0] + \xi^{(1,1)}[x, Y0]\right) + \xi^{(2,0)}[x, Y0]\right) + \\ &\quad Y1\,\eta^{(2,1)}[x, Y0] + Y1\left(Y1\,\eta^{(1,2)}[x, Y0] + \eta^{(2,1)}[x, Y0]\right) + \\ &\quad Y1\left(Y2\,\eta^{(0,2)}[x, Y0] + Y1\,\eta^{(1,2)}[x, Y0] + Y1\left(Y1\,\eta^{(0,3)}[x, Y0] + \eta^{(1,2)}[x, Y0]\right) + \eta^{(2,1)}[x, Y0]\right) + \\ &\quad \eta^{(3,0)}[x, Y0] - Y1\left(Y3\,\xi^{(0,1)}[x, Y0] + Y2\,\xi^{(1,1)}[x, Y0] + 2\,Y2\left(Y1\,\xi^{(0,2)}[x, Y0] + \xi^{(1,1)}[x, Y0]\right) + \right. \\ &\qquad Y1\,\xi^{(2,1)}[x, Y0] + Y1\left(Y1\,\xi^{(1,2)}[x, Y0] + \xi^{(2,1)}[x, Y0]\right) + Y1\left(Y2\,\xi^{(0,2)}[x, Y0] + \\ &\qquad\quad \left. Y1\,\xi^{(1,2)}[x, Y0] + Y1\left(Y1\,\xi^{(0,3)}[x, Y0] + \xi^{(1,2)}[x, Y0]\right) + \xi^{(2,1)}[x, Y0]\right) + \xi^{(3,0)}[x, Y0]\right)\end{aligned}$$

Our Equation :

```
F = Y3 - Y0 Y2 + Y1^2;
```

Y3-Y0 Y2+Y1^2 means Y3->Y0 Y2-Y1^2

```
X2F = D[F, x] ξ[x, Y0] + D[F, Y0] η[x, Y0] + Eta1 D[F, Y1] + Eta2 D[F, Y2] + Eta3 D[F, Y3]
```

$$\begin{aligned}&-Y2\,\eta[x, Y0] + Y3\,\eta^{(0,1)}[x, Y0] - 3\,Y3\left(Y1\,\xi^{(0,1)}[x, Y0] + \xi^{(1,0)}[x, Y0]\right) + \\ &\quad 2\,Y1\left(Y1\,\eta^{(0,1)}[x, Y0] + \eta^{(1,0)}[x, Y0] - Y1\left(Y1\,\xi^{(0,1)}[x, Y0] + \xi^{(1,0)}[x, Y0]\right)\right) + \\ &\quad Y2\,\eta^{(1,1)}[x, Y0] + 2\,Y2\left(Y1\,\eta^{(0,2)}[x, Y0] + \eta^{(1,1)}[x, Y0]\right) - \\ &\quad 3\,Y2\left(Y2\,\xi^{(0,1)}[x, Y0] + Y1\,\xi^{(1,1)}[x, Y0] + Y1\left(Y1\,\xi^{(0,2)}[x, Y0] + \xi^{(1,1)}[x, Y0]\right) + \xi^{(2,0)}[x, Y0]\right) - \\ &\quad Y0\left(Y2\,\eta^{(0,1)}[x, Y0] - 2\,Y2\left(Y1\,\xi^{(0,1)}[x, Y0] + \xi^{(1,0)}[x, Y0]\right) + Y1\,\eta^{(1,1)}[x, Y0] + \right. \\ &\qquad Y1\left(Y1\,\eta^{(0,2)}[x, Y0] + \eta^{(1,1)}[x, Y0]\right) + \eta^{(2,0)}[x, Y0] - Y1\left(Y2\,\xi^{(0,1)}[x, Y0] + \right. \\ &\qquad\quad \left.\left. Y1\,\xi^{(1,1)}[x, Y0] + Y1\left(Y1\,\xi^{(0,2)}[x, Y0] + \xi^{(1,1)}[x, Y0]\right) + \xi^{(2,0)}[x, Y0]\right)\right) + \\ &\quad Y1\,\eta^{(2,1)}[x, Y0] + Y1\left(Y1\,\eta^{(1,2)}[x, Y0] + \eta^{(2,1)}[x, Y0]\right) + \\ &\quad Y1\left(Y2\,\eta^{(0,2)}[x, Y0] + Y1\,\eta^{(1,2)}[x, Y0] + Y1\left(Y1\,\eta^{(0,3)}[x, Y0] + \eta^{(1,2)}[x, Y0]\right) + \eta^{(2,1)}[x, Y0]\right) + \\ &\quad \eta^{(3,0)}[x, Y0] - Y1\left(Y3\,\xi^{(0,1)}[x, Y0] + Y2\,\xi^{(1,1)}[x, Y0] + 2\,Y2\left(Y1\,\xi^{(0,2)}[x, Y0] + \xi^{(1,1)}[x, Y0]\right) + \right. \\ &\qquad Y1\,\xi^{(2,1)}[x, Y0] + Y1\left(Y1\,\xi^{(1,2)}[x, Y0] + \xi^{(2,1)}[x, Y0]\right) + Y1\left(Y2\,\xi^{(0,2)}[x, Y0] + \\ &\qquad\quad \left. Y1\,\xi^{(1,2)}[x, Y0] + Y1\left(Y1\,\xi^{(0,3)}[x, Y0] + \xi^{(1,2)}[x, Y0]\right) + \xi^{(2,1)}[x, Y0]\right) + \xi^{(3,0)}[x, Y0]\right)\end{aligned}$$

```
SymmCond = X2F /. Y3 → Y0 Y2 – Y1^2
```

$$-\text{Y2}\,\eta[\text{x, Y0}] + \left(-\text{Y1}^2 + \text{Y0 Y2}\right)\eta^{(0,1)}[\text{x, Y0}] - 3\left(-\text{Y1}^2 + \text{Y0 Y2}\right)\left(\text{Y1}\,\xi^{(0,1)}[\text{x, Y0}] + \xi^{(1,0)}[\text{x, Y0}]\right) +$$
$$2\,\text{Y1}\left(\text{Y1}\,\eta^{(0,1)}[\text{x, Y0}] + \eta^{(1,0)}[\text{x, Y0}] - \text{Y1}\left(\text{Y1}\,\xi^{(0,1)}[\text{x, Y0}] + \xi^{(1,0)}[\text{x, Y0}]\right)\right) +$$
$$\text{Y2}\,\eta^{(1,1)}[\text{x, Y0}] + 2\,\text{Y2}\left(\text{Y1}\,\eta^{(0,2)}[\text{x, Y0}] + \eta^{(1,1)}[\text{x, Y0}]\right) -$$
$$3\,\text{Y2}\left(\text{Y2}\,\xi^{(0,1)}[\text{x, Y0}] + \text{Y1}\,\xi^{(1,1)}[\text{x, Y0}] + \text{Y1}\left(\text{Y1}\,\xi^{(0,2)}[\text{x, Y0}] + \xi^{(1,1)}[\text{x, Y0}]\right) + \xi^{(2,0)}[\text{x, Y0}]\right) -$$
$$\text{Y0}\left(\text{Y2}\,\eta^{(0,1)}[\text{x, Y0}] - 2\,\text{Y2}\left(\text{Y1}\,\xi^{(0,1)}[\text{x, Y0}] + \xi^{(1,0)}[\text{x, Y0}]\right) + \text{Y1}\,\eta^{(1,1)}[\text{x, Y0}] +\right.$$
$$\text{Y1}\left(\text{Y1}\,\eta^{(0,2)}[\text{x, Y0}] + \eta^{(1,1)}[\text{x, Y0}]\right) + \eta^{(2,0)}[\text{x, Y0}] - \text{Y1}\left(\text{Y2}\,\xi^{(0,1)}[\text{x, Y0}] +\right.$$
$$\left.\left.\text{Y1}\,\xi^{(1,1)}[\text{x, Y0}] + \text{Y1}\left(\text{Y1}\,\xi^{(0,2)}[\text{x, Y0}] + \xi^{(1,1)}[\text{x, Y0}]\right) + \xi^{(2,0)}[\text{x, Y0}]\right)\right) +$$
$$\text{Y1}\,\eta^{(2,1)}[\text{x, Y0}] + \text{Y1}\left(\text{Y1}\,\eta^{(1,2)}[\text{x, Y0}] + \eta^{(2,1)}[\text{x, Y0}]\right) +$$
$$\text{Y1}\left(\text{Y2}\,\eta^{(0,2)}[\text{x, Y0}] + \text{Y1}\,\eta^{(1,2)}[\text{x, Y0}] + \text{Y1}\left(\text{Y1}\,\eta^{(0,3)}[\text{x, Y0}] + \eta^{(1,2)}[\text{x, Y0}]\right) + \eta^{(2,1)}[\text{x, Y0}]\right) +$$
$$\eta^{(3,0)}[\text{x, Y0}] -$$
$$\text{Y1}\left(\left(-\text{Y1}^2 + \text{Y0 Y2}\right)\xi^{(0,1)}[\text{x, Y0}] + \text{Y2}\,\xi^{(1,1)}[\text{x, Y0}] + 2\,\text{Y2}\left(\text{Y1}\,\xi^{(0,2)}[\text{x, Y0}] + \xi^{(1,1)}[\text{x, Y0}]\right) +\right.$$
$$\text{Y1}\,\xi^{(2,1)}[\text{x, Y0}] + \text{Y1}\left(\text{Y1}\,\xi^{(1,2)}[\text{x, Y0}] + \xi^{(2,1)}[\text{x, Y0}]\right) + \text{Y1}\left(\text{Y2}\,\xi^{(0,2)}[\text{x, Y0}] +\right.$$
$$\left.\left.\left.\text{Y1}\,\xi^{(1,2)}[\text{x, Y0}] + \text{Y1}\left(\text{Y1}\,\xi^{(0,3)}[\text{x, Y0}] + \xi^{(1,2)}[\text{x, Y0}]\right) + \xi^{(2,1)}[\text{x, Y0}]\right) + \xi^{(3,0)}[\text{x, Y0}]\right)\right)$$

```
SymmCond = Collect[SymmCond, {Y1, Y2}]
```

$$-3\,\text{Y2}^2\,\xi^{(0,1)}[\text{x, Y0}] - \text{Y1}^4\,\xi^{(0,3)}[\text{x, Y0}] +$$
$$\text{Y1}^3\left(2\,\xi^{(0,1)}[\text{x, Y0}] + \text{Y0}\,\xi^{(0,2)}[\text{x, Y0}] + \eta^{(0,3)}[\text{x, Y0}] - 3\,\xi^{(1,2)}[\text{x, Y0}]\right) -$$
$$\text{Y0}\,\eta^{(2,0)}[\text{x, Y0}] + \text{Y2}\left(-\eta[\text{x, Y0}] - \text{Y0}\,\xi^{(1,0)}[\text{x, Y0}] + 3\,\eta^{(1,1)}[\text{x, Y0}] - 3\,\xi^{(2,0)}[\text{x, Y0}]\right) +$$
$$\text{Y1}^2\left(\eta^{(0,1)}[\text{x, Y0}] - \text{Y0}\,\eta^{(0,2)}[\text{x, Y0}] - 6\,\text{Y2}\,\xi^{(0,2)}[\text{x, Y0}] + \xi^{(1,0)}[\text{x, Y0}] + 2\,\text{Y0}\,\xi^{(1,1)}[\text{x, Y0}] +\right.$$
$$3\,\eta^{(1,2)}[\text{x, Y0}] - 3\,\xi^{(2,1)}[\text{x, Y0}]\Big) + \eta^{(3,0)}[\text{x, Y0}] + \text{Y1}\left(2\,\eta^{(1,0)}[\text{x, Y0}] -\right.$$
$$2\,\text{Y0}\,\eta^{(1,1)}[\text{x, Y0}] + \text{Y2}\left(-\text{Y0}\,\xi^{(0,1)}[\text{x, Y0}] + 3\,\eta^{(0,2)}[\text{x, Y0}] - 9\,\xi^{(1,1)}[\text{x, Y0}]\right) +$$
$$\left.\text{Y0}\,\xi^{(2,0)}[\text{x, Y0}] + 3\,\eta^{(2,1)}[\text{x, Y0}] - \xi^{(3,0)}[\text{x, Y0}]\right)$$

```
eq1 = Coefficient[SymmCond, Y2^2]
```

$$-3\,\xi^{(0,1)}[\text{x, Y0}]$$

```
DSolve[eq1 == 0, ξ, {x, Y0}]
```

$$\{\{\xi \to \text{Function}[\{\text{x, Y0}\}, \text{C}[1][\text{x}]]\}\}$$

```
ξ[x_, Y0_] = a1[x]
```

$$\text{a1}[\text{x}]$$

```
SymmCond = Collect[SymmCond, {Y1, Y2}]
```

$$\text{Y1}^3\,\eta^{(0,3)}[\text{x, Y0}] + \text{Y2}\left(-\eta[\text{x, Y0}] - \text{Y0}\,\text{a1}'[\text{x}] - 3\,\text{a1}''[\text{x}] + 3\,\eta^{(1,1)}[\text{x, Y0}]\right) +$$
$$\text{Y1}^2\left(\text{a1}'[\text{x}] + \eta^{(0,1)}[\text{x, Y0}] - \text{Y0}\,\eta^{(0,2)}[\text{x, Y0}] + 3\,\eta^{(1,2)}[\text{x, Y0}]\right) -$$
$$\text{Y0}\,\eta^{(2,0)}[\text{x, Y0}] + \text{Y1}\left(\text{Y0}\,\text{a1}''[\text{x}] - \text{a1}^{(3)}[\text{x}] + 3\,\text{Y2}\,\eta^{(0,2)}[\text{x, Y0}] +\right.$$
$$\left.2\,\eta^{(1,0)}[\text{x, Y0}] - 2\,\text{Y0}\,\eta^{(1,1)}[\text{x, Y0}] + 3\,\eta^{(2,1)}[\text{x, Y0}]\right) + \eta^{(3,0)}[\text{x, Y0}]$$

```
eq2 = Coefficient[SymmCond, Y1^3]
```

$$\eta^{(0,3)}[\text{x, Y0}]$$

```
DSolve[eq2 == 0, η, {x, Y0}]
```

$$\left\{\left\{\eta \to \text{Function}\left[\{\text{x, Y0}\}, \text{C}[1][\text{x}] + \text{Y0}\,\text{C}[2][\text{x}] + \text{Y0}^2\,\text{C}[3][\text{x}]\right]\right\}\right\}$$

```
η[x_, Y0_] = b1[x] + Y0 b2[x] + Y0^2 b3[x]
```

```
b1[x] + Y0 b2[x] + Y0^2 b3[x]
```

```
{ξ[x, y], η[x, y]}
```

$$\{a1[x],\ b1[x] + y\, b2[x] + y^2\, b3[x]\}$$

```
SymmCond = Collect[SymmCond, {Y0, Y1, Y2}]
```

$$Y1^2\,(b2[x] + a1'[x] + 6\,b3'[x]) + Y2\,(-b1[x] + 3\,b2'[x] - 3\,a1''[x]) - Y0^3\,b3''[x] + Y1\,\left(6\,Y2\,b3[x] + 2\,b1'[x] + 3\,b2''[x] - a1^{(3)}[x]\right) + b1^{(3)}[x] + Y0\,\left(Y2\,(-b2[x] - a1'[x] + 6\,b3'[x]) - b1''[x] + Y1\,(a1''[x] + 6\,b3''[x]) + b2^{(3)}[x]\right) + Y0^2\,\left(-Y2\,b3[x] - 2\,Y1\,b3'[x] - b2''[x] + b3^{(3)}[x]\right)$$

```
eq3 = Coefficient[SymmCond, Y0^3]
```

$$-b3''[x]$$

```
b3[x_] = c1 + c2 x;
```

```
SymmCond = Collect[SymmCond, {Y0, Y1, Y2}]
```

$$Y1^2\,(6\,c2 + b2[x] + a1'[x]) + Y2\,(-b1[x] + 3\,b2'[x] - 3\,a1''[x]) + Y0^2\,(-2\,c2\,Y1 + (-c1 - c2\,x)\,Y2 - b2''[x]) + Y1\,\left(6\,(c1 + c2\,x)\,Y2 + 2\,b1'[x] + 3\,b2''[x] - a1^{(3)}[x]\right) + b1^{(3)}[x] + Y0\,\left(Y2\,(6\,c2 - b2[x] - a1'[x]) + Y1\,a1''[x] - b1''[x] + b2^{(3)}[x]\right)$$

```
eq3 = Coefficient[SymmCond, Y0^2]
```

$$-2\,c2\,Y1 + (-c1 - c2\,x)\,Y2 - b2''[x]$$

```
c2 = 0; c1 = 0;
b2[x_] = c3 x + c4;
```

```
SymmCond = Collect[SymmCond, {Y0, Y1, Y2}]
```

$$Y1^2\,(c4 + c3\,x + a1'[x]) + Y2\,(3\,c3 - b1[x] - 3\,a1''[x]) + Y0\,(Y2\,(-c4 - c3\,x - a1'[x]) + Y1\,a1''[x] - b1''[x]) + Y1\,\left(2\,b1'[x] - a1^{(3)}[x]\right) + b1^{(3)}[x]$$

```
eq3 = Coefficient[SymmCond, Y1^2]
```

$$c4 + c3\,x + a1'[x]$$

```
DSolve[eq3 == 0, a1, x]
```

$$\left\{\left\{a1 \to \text{Function}\left[\{x\},\ -c4\,x - \frac{c3\,x^2}{2} + C[1]\right]\right\}\right\}$$

```
a1[x_] = -c4 x - c3 x^2/2 + c5;
```

```
SymmCond = Collect[SymmCond, {Y0, Y1, Y2}]
```

$$Y2\,(6\,c3 - b1[x]) + 2\,Y1\,b1'[x] + Y0\,(-c3\,Y1 - b1''[x]) + b1^{(3)}[x]$$

```
b1[x_] = 6 c3;
```

```
SymmCond = Collect[SymmCond, {Y0, Y1, Y2}]
```

```
- c3 Y0 Y1
```

```
c3 = 0;
```

```
{ξ[x, y], η[x, y]}
```

```
{c5 - c4 x, c4 y}
```

```
{ξ[x, y], η[x, y]} /. {c4 → 1, c5 → 0}
{ξ[x, y], η[x, y]} /. {c4 → 0, c5 → 1}
```

```
{-x, y}
```

```
{1, 0}
```