On the construction of harmonic functions with rational degrees of homogeneity in Euclidean space

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Structure of the presentation

- **Part 1:** Some operators that we used to construct homogeneous harmonic functions;
- **Part 2:** Composition of these operators;
- **Part 3:** Inverse operators and examples of using them.

Definitions

Harmonic function: $\Delta u = 0$ (Laplace equation);
The Laplacian operator: $\Delta = \sum_{i=1}^{n} \partial^2_{x_i x_i}$, where n - quantity of dimensions in the space.

Homogeneous function according to Euler: $u(\lambda x) = \lambda^q u(x)$, where q is degree of homogeneity of the function.

Application: Quantum field theory.

Previous investigations:

- three-dimensional space;
- integer degree of homogeneity;
- degree of homogeneity is $q \in (-4; 0)$.

Purpose: Construction of harmonic functions in two-dimensional Euclidean space with arbitrary non-integer degree of homogeneity.

Suppose G be a domain in $\mathbb{R}^2$ (simply- or multi-connected).

Let us introduce the space
$\mathbf{H}(G) = \{u \in C^2(G) : \Delta u = 0, u(\lambda x) = \lambda^q u(x), q \in \mathbb{Q}, x \in G\}$.

It is required to construct 'new' functions $v \in \mathbf{H}(G)$ by a given initial function $u \in \mathbf{H}(G)$.

Some methods of construction of homogeneous harmonic functions

1. By Thomson transform of the original function

$$v(z,\bar{z}) = \mathcal{T}u(z,\bar{z}) = u(\tfrac{1}{\bar{z}}, \tfrac{1}{z})$$

2. Using the integral operator

$$v(z,\bar{z}) = \mathcal{K}u(z,\bar{z}) = i\int_{z_0}^{z}(u'_{\bar{z}}d\bar{z} - u'_z dz)$$

3. By applying first order differential operators $\{\mathcal{L}\}$

$$v(z,\bar{z}) = \mathcal{L}u(z,\bar{z})$$

$$\mathcal{L} = (\alpha_1 z + \beta_1)\partial_z + (\alpha_2\bar{z} + \beta_2)\partial_{\bar{z}}, \quad \text{where } \alpha_k, \beta_k \in \mathbb{C}, k = 1, 2$$

Thomson transform $\mathcal{T}$

Harmonic functions $u \in \mathbf{H}(G)$ satisfy the Kelvin-Thomson transformation:

$$v(z,\bar{z}) = \mathcal{T}u(z,\bar{z}) = u(\tfrac{1}{\bar{z}}, \tfrac{1}{z})$$

- guarantees membership $v \in \mathbf{H}(G)$
- changes the degree of homogeneity of the function

Action of the Thomson transform on the function

$$u(z,\bar{z}) = \mu z^q + \gamma \bar{z}^q, u \in \mathbf{H}(G)$$

leads to the function

$$v(z,\bar{z}) = \gamma \bar{z}^{-q} + \mu z^{-q}.$$

$\implies$ $v \in \mathbf{H}(G)$ **and the degree of homogeneity changed by taking the value of -q.**

Integral operator $\mathcal{K}$

$\omega = i(u'_{\bar{z}} d_{\bar{z}} - u'_z d_z)$ - closed differential form.

$$v(z,\bar{z}) = \mathcal{K}u(z,\bar{z}) = i \int_{z_0}^{z} (u'_{\bar{z}} d\bar{z} - u'_z dz)$$

- guarantees membership $v \in \mathbf{H}(G)$;
- degree of homogeneity q is preserved $v(\lambda z, \lambda \bar{z}) = \lambda^q v(z,\bar{z})$.

As a function we take a polynomial with degree of homogeneity q

$$u = \mu z^q + \gamma \bar{z}^q, u \in \mathbf{H}(G)$$

'New' homogeneous harmonic function is given by

$$v(z,\bar{z}) = i[\gamma \bar{z}^q - \mu z^q] - i[\gamma \bar{z}_0^q - \mu z_0^q].$$

$\Longrightarrow$ **degree of homogeneity is equal to the degree of the original function.**

Differential operators $\mathcal{L}$

In general the operators will be given by:

$$\mathcal{L}_{\mathrm{A}} = (\alpha_1 z + \beta_1)\partial_z + (\alpha_2 \bar{z} + \beta_2)\partial_{\bar{z}} \qquad ,$$

$$\text{where } \mathrm{A}(z,\bar{z}) = (\alpha_1 z + \beta_1, \alpha_2 \bar{z} + \beta_2), \quad \alpha_k, \beta_k \in \mathbb{C}, k = 1, 2$$

Action of the operator $\mathcal{L}_{\mathrm{A}}$ on the function $u = \mu z^q + \gamma \bar{z}^q$ is given by:

$$v(z,\bar{z}) = \mathcal{L}_{\mathrm{A}} u = q\mu(\alpha_1 z + \beta_1) z^{q-1} + q\gamma(\alpha_2 \bar{z} + \beta_2)\bar{z}^{q-1}$$

To ensure the harmonicity of the function, let us assume

$$\alpha_1 = \alpha_2 = 0 : v_1(z,\bar{z}) = q[\mu\beta_1 z^{q-1} + \gamma\beta_2 \bar{z}^{q-1}]$$
$$\beta_1 = \beta_2 = 0 : v_2(z,\bar{z}) = q[\mu\alpha_1 z^{q} + \gamma\alpha_2 \bar{z}^{q}]$$

$\Longrightarrow$ **the degree of homogeneity is either maintained or reduced by one unit.**

Composition of transformations $\mathcal{T}$ и $\mathcal{L}_{\mathrm{A}}$

Consider the action of the operator $\mathcal{L}_{\mathrm{A}} \circ \mathcal{T}$ on the harmonic function $u(z, \bar{z})$:

$$v_1(z,\bar{z}) = \mathcal{L}_1 u(z,\bar{z}) = (\mathcal{L}_{\mathrm{A}} \circ \mathcal{T}) u(z,\bar{z}) = -(\tfrac{\alpha_1}{z} + \tfrac{\beta_1}{z^2}) u'_z(\tfrac{1}{\bar{z}}, \tfrac{1}{z}) - (\tfrac{\alpha_2}{\bar{z}} + \tfrac{\beta_2}{\bar{z}^2}) u'_{\bar{z}}(\tfrac{1}{\bar{z}}, \tfrac{1}{z})$$

$\Longrightarrow$ **Function** $v_1 \in \mathbf{H}(G)$ **either at** $\alpha_1 = \alpha_2 = 0,$ **or at** $\beta_1 = \beta_2 = 0$

Consider the action of the operator $\mathcal{T} \circ \mathcal{L}_{\mathrm{A}}$ on the harmonic function $u(z, \bar{z})$:

$$v_2(z,\bar{z}) = \mathcal{L}_2 u(z,\bar{z}) = (\mathcal{T} \circ \mathcal{L}_{\mathrm{A}}) u(z,\bar{z}) = (\tfrac{\alpha_1}{\bar{z}} + \beta_1) u'_z(\tfrac{1}{\bar{z}}, \tfrac{1}{z}) + (\tfrac{\alpha_2}{z} + \beta_2) u'_{\bar{z}}(\tfrac{1}{\bar{z}}, \tfrac{1}{z})$$

$\Longrightarrow$ **Function** $v_2 \in \mathbf{H}(G)$ **at** $\alpha_1 = \alpha_2 = 0.$

Reconstruction formula

Consider the action of the composition of transformations $\mathcal{L}_{\mathrm{A}} \circ \mathcal{T}$ on the function $u(z, \bar{z})$:

$$w(z, \bar{z}) = \mathcal{L}_1 u(z, \bar{z}) = \mathcal{L}_{\mathrm{A}} v(z, \bar{z})$$

Let us fix the initial conditions $u(\frac{1}{\bar{z}(0)}, \frac{1}{z(0)}) = v(z(0), \bar{z}(0)) = v(z_0, \bar{z}_0) = v_0$.

Function v will be the solution of following Cauchy problem:

$$\mathcal{L}_{\mathrm{A}} v(z, \bar{z}) = w; \quad v(z_0, \bar{z}_0) = v_0$$

The above problem has a solution in the form:

$$u(z, \bar{z}) = \mathcal{T}^{-1} u(\tfrac{1}{\bar{z}}, \tfrac{1}{z}) = u_0 + \mathcal{T}^{-1} \int_0^t w(z(\tau), \bar{z}(\tau)) d\tau$$

Case №1

In the vector field $\mathrm{A}(z,\bar{z}) = (\alpha_1 z + \beta_1, \alpha_2 \bar{z} + \beta_2)$ we assume the parameters $\beta_1 = \beta_2 = 0$.

The field taking into account the initial conditions will be given by $\mathrm{A}_1(z,\bar{z}) = (\alpha_1 z, \alpha_2 \bar{z})$;

$$\begin{cases} \dot{z} = \alpha_1 z \\ \dot{\bar{z}} = \alpha_2 \bar{z} \\ z(0) = z_0 \\ \bar{z}(0) = \bar{z}_0 \end{cases} \implies \quad z(t) = z_0 e^{\alpha_1 t}, \quad \bar{z}(t) = \bar{z}_0 e^{\alpha_2 t}\ ;$$

$$u(z,\bar{z}) = \mathcal{T}^{-1} u(\tfrac{1}{\bar{\bar{z}}}, \tfrac{1}{z}) = u_0 + \mathcal{T}^{-1} \int\limits_0^t w(z_0 e^{\alpha_1 \tau}, \bar{z}_0 e^{\alpha_2 \tau}) d\tau$$

Example №1

Consider a homogeneous function of the following form:

$$w = \mu z^q + \gamma \bar{z}^q \text{ , } w \in \mathbf{H}(G); \mu, \gamma \in \mathbb{C}$$

The solution is of the form:

$$u(z(t), \bar{z}(t)) = u_0 + \frac{\mu}{\alpha_1 q} \bar{z}^{-q} + \frac{\gamma}{\alpha_2 q} z^{-q} + C_0, \quad C_0 = -\frac{\mu z_0^q}{\alpha_1 q} - \frac{\gamma \bar{z}_0^q}{\alpha_2 q}$$

$\Longrightarrow$ function $u \in \mathbf{H}(G)$ under the condition $u_0 + C_0 = 0.$

Case №2

In the vector field $\mathrm{A}(z,\bar{z}) = (\alpha_1 z + \beta_1, \alpha_2 \bar{z} + \beta_2)$ we assume the parameters $\alpha_1 = \alpha_2 = 0$.

The field taking into account the initial conditions will be given by $\mathrm{A}_2(z,\bar{z}) = (\beta_1, \beta_2)$;

$$\begin{cases} \dot{z} = \beta_1 \\ \dot{\bar{z}} = \beta_2 \\ z(0) = z_0 \\ \bar{z}(0) = \bar{z}_0 \end{cases} \implies \quad z(t) = \beta_1 t + z_0, \quad \bar{z}(t) = \beta_2 t + \bar{z}_0 \ ;$$

$$u(z,\bar{z}) = \mathcal{T}^{-1} u(\tfrac{1}{\bar{z}}, \tfrac{1}{z}) = u_0 + \mathcal{T}^{-1} \int\limits_0^t w(\beta_1 \tau + z_0, \beta_2 \tau + \bar{z}_0) d\tau$$

Example №2

Consider a homogeneous function of the following form:

$$w = \mu z^q + \gamma \bar{z}^q \text{ , } w \in \mathbf{H}(G); \mu, \gamma \in \mathbb{C}$$

The solution is of the form:

$$u(z, \bar{z}) = u_0 + \frac{\mu}{\beta_1(q+1)} \bar{z}^{-q-1} + \frac{\gamma}{\beta_2(q+1)} z^{-q-1} + C_0, \quad C_0 = -\frac{\mu}{\beta_1(q+1)} z_0^{q+1} - \frac{\gamma}{\beta_2(q+1)} \bar{z}_0^{q+1}$$

$\implies$ function u $\in \mathbf{H}(G)$ under the condition $u_0 + C_0 = 0$.

Results

- The transformations in the space of homogeneous harmonic functions, allowing to construct harmonic functions with a degree of homogeneity different from the initial one, are specified;
- The solution of the problem on construction of homogeneous harmonic functions on initial ones in a general form for the specified methods is received:
- Some examples are given to illustrate these methods.

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